ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$\int_0^\infty\int_0^\infty e^{-x}\left(\frac{x}{y}\right)^2\ln\left(1+\frac{y^2}{x^2}\right)\ln(y)\,dxdy=\pi(2-\gamma)$$

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$$\Omega=\int_0^\infty\int_0^\infty e^{-x}\left(\frac{x}{y}\right)^2\ln\left(1+\frac{y^2}{x^2}\right)\ln(y)\,dxdy,\left\{\frac{y}{x}=t,dt=dy,t[\infty;0]\right\}$$

$$\Omega=\int_0^\infty\int_0^\infty\frac{e^{-x}x\ln(1+t^2)\ln(xt)}{t^2}dxdt=$$

$$=\int_0^\infty e^{-x}x\ln(x)\int_0^\infty\frac{\ln(1+t^2)}{t^2}dxdt+\int_0^\infty e^{-x}x\int_0^\infty\frac{\ln(t)\ln(1+t^2)}{t^2}dxdt=$$

$$=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^\infty e^{-x}x\ln(x)\int_0^\infty\frac{\ln(1+t^2)}{t^2}dxdt=I\bullet J$$

$$I=\int_0^\infty e^{-x}x\ln(x)\,dx=\lim_{a\to1}\frac{d}{da}\int_0^\infty x^a e^{-x}dx=\lim_{a\to1}\frac{d}{da}\Gamma(a+1)=$$

$$=\lim_{a\to1}\psi^{(0)}(a+1)\Gamma(a+1)=\psi^{(0)}(2)=1-\gamma$$

$$note:\left\{Gamma\ function:\ \int_0^\infty e^{-x}x^{a-1}dx=\Gamma(a)\right\}$$

$$J=\int_0^\infty\frac{\ln(1+t^2)}{t^2}dt=\int_0^1\frac{\ln(1+t^2)}{t^2}dt+\int_1^\infty\frac{\ln(1+t^2)}{t^2}dt=J_1+J_2$$

$$J_1=\int_0^1\frac{\ln(1+t^2)}{t^2}dt=-\sum_{n=1}^\infty\frac{(-1)^n}{n}\int_0^1 t^{2n-2}\,dt=-\sum_{n=1}^\infty\frac{(-1)^n}{n(2n-1)}=$$

$$=\sum_{n=1}^\infty\frac{(-1)^n}{n}-2\sum_{n=1}^\infty\frac{(-1)^n}{2n-1}=\frac{\pi}{2}-\ln(2)$$

$$J_2=\int_1^\infty\frac{\ln(1+t^2)}{t^2}dt,\qquad\left\{\frac{1}{t}=u,du=-u^2dt,u[0;1]\right\}$$

$$J_2=\int_0^1\ln(1+u^2)du-2\int_0^1\ln(u)\,du=-\sum_{n=1}^\infty\frac{(-1)^n}{n}\int_0^1 u^{2n}du-2[u\ln(u)-u]_0^1=$$

$$=-\sum_{n=1}^\infty\frac{(-1)^n}{n(2n+1)}+2=2\sum_{n=1}^\infty\frac{(-1)^n}{2n+1}-\sum_{n=1}^\infty\frac{(-1)^n}{n}+2=\frac{\pi}{2}-2+\ln(2)+2=\frac{\pi}{2}+\ln(2)$$

$$J=J_1+J_2=\frac{\pi}{2}-\ln(2)+\frac{\pi}{2}+\ln(2)=\pi,\qquad\Omega_1=J\bullet I=\pi(1-\gamma)$$

$$\Omega_2=\int_0^\infty e^{-x}x\int_0^\infty\frac{\ln(t)\ln(1+t^2)}{t^2}dtdx=M\bullet K$$

$$M=\int_0^\infty e^{-x}xdx=\Gamma(2)=1$$

$$K=\int_0^\infty\frac{\ln(t)\ln(1+t^2)}{t^2}dt=\int_0^1\frac{\ln(t)\ln(1+t^2)}{t^2}dt+\int_1^\infty\frac{\ln(t)\ln(1+t^2)}{t^2}dt=K_1+K_2$$

$$K_1=\int_0^1\frac{\ln(t)\ln(1+t^2)}{t^2}dt=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 t^{2n-2}\ln(t)\,dt=\sum_{n=1}^{\infty}\frac{(-1)^n}{n(2n-1)^2}=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}-2\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}=\frac{\pi}{2}-2G-\ln(2)$$

$$K_2=\int_1^{\infty}\frac{\ln(t)\ln(1+t^2)}{t^2}dt,\left\{\frac{1}{t}=u,du=-u^2dt,u[0;1]\right\}$$

$$K_2=-\int_0^1\ln\left(1+\frac{1}{u^2}\right)\ln(u)\,du=-\int_0^1\ln(u)\ln(1+u^2)du+2\int_0^1 ln^2(u)du=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 u^{2n}\ln(u)\,du+2\left[2u+uln^2(u)-2uln(u)\right]_0^1=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(2n+1)^2}+4=$$

$$=2\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+4=2G-4+\frac{\pi}{2}+\ln(2)+4=$$

$$=2G+\frac{\pi}{2}+\ln(2)$$

$$K=K_1+K_2=\frac{\pi}{2}-2G-\ln(2)+2G+\frac{\pi}{2}+\ln(2)=\pi$$

$$\Omega_2=K\bullet M=\pi$$

$$\Omega=\Omega_1+\Omega_2=\pi+\pi(1-\gamma)=\pi(2-\gamma)$$

$$\int_0^{\infty}\int_0^{\infty}e^{-x}\left(\frac{x}{y}\right)^2\ln\left(1+\frac{y^2}{x^2}\right)\ln(y)dxdy=\pi(2-\gamma)$$