ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int\int_{[0;1]^2} \ln\left(\frac{1}{e^x+e^y}\right)dxdy = 2Li_3(-e)+\frac{3\zeta(3)}{2}+\frac{\pi^2}{6}-\frac{1}{3}$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \int\int_{[0;1]^2} \ln\left(\frac{1}{e^x+e^y}\right)dxdy = \int\int_{[0;1]^2} \ln\left(\frac{e^{-x}}{1+e^{y-x}}\right)dxdy =$$

$$= \int\int_{[0;1]^2} \ln(e^{-x})dxdy - \int\int_{[0;1]^2} \ln(1+e^{y-x})\,dxdy =$$

$$= -\int\int_{[0;1]^2} xdxdy + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int\int_{[0;1]^2}\frac{e^{ny}}{e^{nx}}dxdy =$$

$$= -\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_{[0;1]} e^{ny}\left[\frac{e^{-nx}}{n}\right]_0^1 dy = -\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{e^{-n}}{n}-\frac{1}{n}\right)\int_{[0;1]} e^{ny}dy =$$

$$= -\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{e^{-n}}{n}-\frac{1}{n}\right)\left[\frac{e^{ny}}{n}\right]_0^1 = -\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{e^{-n}}{n}-\frac{1}{n}\right)\left(\frac{e^{n}}{n}-\frac{1}{n}\right) =$$

$$= -\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{2}{n^2}-\frac{e^{-n}}{n^2}-\frac{e^{n}}{n^2}\right) =$$

$$= -\frac{1}{2}-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}+\sum_{n=1}^{\infty}\frac{(-1)^n e^{-n}}{n^3}+\sum_{n=1}^{\infty}\frac{(-1)e^{n}}{n^3} =$$

$$= -\frac{1}{2}+2\eta(3)+Li_3\left(-\frac{1}{e}\right)+Li_3(-e) = 2Li_3(-e)+\frac{3\zeta(3)}{2}+\frac{\pi^2}{6}-\frac{1}{3}$$

$$\textit{notes}:\left\{\begin{array}{c} Li_s(z)+(-1)^s Li_s\left(\frac{1}{z}\right) = \frac{(2\pi i)^s}{\Gamma(s)}\zeta\left(1-s;\frac{1}{2}+\frac{\ln(-z)}{2\pi i}\right) \\ Li_3(-z)-Li_3\left(-\frac{1}{z}\right) = -\frac{1}{6}\ln^3(z)-\zeta(2)\ln(z);\ Li_3(-e)-Li_3\left(-\frac{1}{e}\right) = -\frac{1}{6}-\zeta(2) \\ \eta(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^z} = (1-2^{1-z})\zeta(z) \end{array}\right\}$$