ROMANIAN MATHEMATICAL MAGAZINE

If:

$$\Omega=\int_0^\infty \frac{\ln(1+x^2)\ln(x)}{x^2}dx+\frac{1}{4}\int_0^\infty \frac{\ln\left(1+\frac{y^2}{4}\right)\ln(y)}{y^2}dy+\frac{1}{9}\int_0^\infty \frac{\ln\left(1+\frac{z^2}{9}\right)\ln(z)}{z^2}dz+\cdots$$

$$\textit{Then, show that: } \Omega=\pi\big(\zeta(3)-\zeta'(3)\big)$$

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$$\Omega=\sum_{n=1}^\infty \frac{1}{n^2}\int_0^\infty \frac{\ln\left(1+\frac{x^2}{n^2}\right)\ln(x)}{x^2}dx=\sum_{n=1}^\infty \frac{1}{n^2}f\left(\frac{1}{n}\right)$$

$$f\left(\frac{1}{n}\right)=\int_0^\infty \frac{\ln\left(1+\frac{x^2}{n^2}\right)\ln(x)}{x^2}dx,\left\{\frac{1}{n}=a\right\}\to f(a)=\int_0^\infty \frac{\ln(1+a^2x^2)\ln(x)}{x^2}dx$$

$$\frac{d}{da}f(a)=2a\int_0^\infty \frac{\ln(x)}{(1+a^2x^2)}dx;\left\{\tan^{-1}(ax)=t;\frac{dt}{dx}=\frac{a}{1+a^2x^2};\ t\left[\frac{\pi}{2};0\right]\right\}$$

$$\frac{d}{da}f(a)=2\int_0^{\frac{\pi}{2}}\ln\left(\frac{\tan(t)}{a}\right)dt=2\int_0^{\frac{\pi}{2}}\ln\big(\tan(t)\big)dt-2\ln(a)\int_0^{\frac{\pi}{2}}dt=2I-\pi\ln(a)$$

$$I=\int_0^{\frac{\pi}{2}}\ln\big(\tan(t)\big)dt,\ \left\{\frac{\pi}{2}-t\to t\right\}\ I=\int_0^{\frac{\pi}{2}}\ln\left(\tan\left(\frac{\pi}{2}-t\right)\right)dt=\int_0^{\frac{\pi}{2}}\ln\big(\cot(t)\big)dt$$

$$I=\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\ln\big(\tan(t)\big)dt+\int_0^{\frac{\pi}{2}}\ln\big(\cot(t)\big)dt\right)=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\big(\tan(t)\cot(t)\big)dt=0$$

$$\frac{d}{da}f(a)=-\pi\ln(a);f(a)=-\pi\int\ln(a)da=\pi a-\pi a\ln(a),\left\{a=\frac{1}{n}\right\}$$

$$f\left(\frac{1}{n}\right)=\frac{\pi}{n}-\frac{\pi}{n}\ln\left(\frac{1}{n}\right)=\pi\left(\frac{\ln(n)}{n}+\frac{1}{n}\right)$$

$$\Omega=\sum_{n=1}^\infty \frac{1}{n^2}f\left(\frac{1}{n}\right)=\pi\sum_{n=1}^\infty \frac{\ln(n)}{n^3}+\pi\sum_{n=1}^\infty \frac{1}{n^3}=\pi\lim_{a\to 0}\frac{d}{da}\sum_{n=1}^\infty \frac{1}{n^{3-a}}+\pi\zeta(3)=$$

$$=\pi\lim_{a\to 0}\frac{d}{da}\zeta(3-n)+\pi\zeta(3)=-\pi\zeta'(3)+\pi\zeta(3)=\pi\big(\zeta(3)-\zeta'(3)\big)$$