ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1\int_0^\infty \frac{x/y}{\sqrt{\sinh\left(\frac{x}{y}\right)}}dxdy = \frac{\pi}{2\sqrt{2}}\varpi$$

where ϖ, is the lemniscate constant .

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$$\Omega = \int_0^1\int_0^\infty \frac{x}{y\sqrt{\sinh\left(\frac{x}{y}\right)}}dxdy;\ subtitation\ \left\{\frac{x}{y} = t, dt = \frac{dx}{y}, t[\infty;0]\right\}$$

$$\Omega = \int_0^1 y\int_0^\infty \frac{t}{\sqrt{\sinh(t)}}dtdy = K\bullet M,\quad K = \int_0^1 ydy = \frac{1}{2},\quad M = \int_0^\infty \frac{t}{\sqrt{\sinh(t)}}dt$$

$$M = \int_0^\infty \frac{t}{\sqrt{\sinh(t)}}dt = \sqrt{2}\int_0^\infty \frac{t}{\sqrt{e^t - e^{-t}}}dt$$

$$subtitation: \{e^{-t} = u, du = -udt;\ u[0;1]\}$$

$$M = -\sqrt{2}\int_0^1 \frac{\sqrt{u}\ln(u)}{u\sqrt{1-u^2}}du,\qquad \{u^2 \to u\}$$

$$M = -\frac{\sqrt{2}}{4}\int_0^1 \frac{\ln(u)}{u^{-\frac{3}{4}}\sqrt{1-u}}du = -\frac{\sqrt{2}}{4}\lim_{a\to-\frac{3}{4}}\frac{d}{da}\int_0^1 \frac{u^a}{(1-u)^{\frac{1}{2}}}du =$$

$$= -\frac{\sqrt{2}}{4}\lim_{a\to-\frac{3}{4}}\frac{d}{da}\beta\left(a+1;\frac{1}{2}\right) = -\frac{\sqrt{2}}{4}\lim_{a\to-\frac{3}{4}}\frac{d}{da}\left(\frac{\Gamma(a+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(a+\frac{3}{2}\right)}\right) =$$

$$= -\frac{\sqrt{2\pi}}{4}\lim_{a\to-\frac{3}{4}}\left(\frac{\Gamma(a+1)\left(\psi^{(0)}(a+1) - \psi^{(0)}\left(a+\frac{3}{2}\right)\right)}{\Gamma\left(a+\frac{3}{2}\right)}\right) =$$

$$= -\frac{\sqrt{2\pi}}{4}\left(\frac{\Gamma\left(\frac{1}{4}\right)\left(\psi^{(0)}\left(\frac{1}{4}\right) - \psi^{(0)}\left(1-\frac{1}{4}\right)\right)}{\Gamma\left(\frac{3}{4}\right)}\right) = \frac{\pi\sqrt{2\pi}}{4}\frac{\Gamma^2\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4}\Gamma^2\left(\frac{1}{4}\right) =$$

$$= \frac{\pi\sqrt{2}}{2}\varpi = \frac{\pi}{\sqrt{2}}\varpi,\qquad \int_0^1\int_0^\infty \frac{\frac{x}{y}}{\sqrt{\sinh\left(\frac{x}{y}\right)}}dxdy = K\bullet M = \frac{\pi}{2\sqrt{2}}\varpi$$

$$notes: \left\{\begin{array}{c} polygamma: (-1)^m\psi^{(m)}(1-z) - \psi^{(m)}(z) = \pi\frac{d^m}{dz^m}\cot(\pi z) \\ gamma: \Gamma(1-z)\Gamma(z) = \pi\csc(\pi z);\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \\ lemniscate\ constant: \varpi = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}} \end{array}\right\}$$