ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1\int_0^1 \frac{e^{x+y}-1}{e^{x+y}+1}dxdy = 4\chi_2(e^2) - 4\mathrm{Li}_2(e) + \frac{\pi^2}{6} - 1$$

where χ_v is the Legendre's chi function, $\mathrm{Li}_2(z)$ is the dilogarithm or Spence's function.

Proposed by Ankush Kumar Parcha-India

Solution by Cosghun Memmedov-Azerbaijan

$$\int_0^1\int_0^1 \frac{e^{x+y}-1}{e^{x+y}+1}dxdy \overset{\{x+y=t\}}{\triangleq} \int_0^1\int_y^{y+1} \frac{e^t-1}{e^t+1}dtdy = \int_0^1\int_y^{y+1} \frac{1-e^{-t}}{1+e^{-t}}dtdy =$$

$$= \int_0^1\int_y^{y+1} dtdy - 2\int_0^1\int_y^{y+1} \frac{e^{-t}}{1+e^{-t}}dtdy = \int_0^1 dy + 2\sum_{n=1}^{\infty}(-1)^n \int_0^1\int_y^{y+1} e^{-tn}dtdy =$$

$$= 1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1\left(e^{-yn} - e^{-(y+1)n}\right)dtdy = 1 + 2\left(-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} - \right.$$

$$\left. 2\sum_{n=1}^{\infty}\frac{(-e^{-1})^n}{n^2} + \sum_{n=1}^{\infty}\frac{(-e^{-2})^n}{n^2}\right) = 1 + 2\left(-\frac{\pi^2}{12} + \mathrm{Li}_2\left(-\frac{1}{e^2}\right) - 2\mathrm{Li}_2\left(-\frac{1}{e}\right)\right) =$$

$$= 1 + 2\left(-\frac{\pi^2}{12} - 2 - \frac{\pi^2}{6} - \mathrm{Li}_2(-e^2) + 1 + \frac{\pi^2}{3} + \mathrm{Li}_2(e^2) - 2\mathrm{Li}_2(e)\right) = 1 + \frac{\pi^2}{6} + 4\chi_2(e^2) - 4\mathrm{Li}_2(e) - 2 =$$

$$= 4\chi_2(e^2) - 4\mathrm{Li}_2(e) + \frac{\pi^2}{6} - 1$$

Notes: Dilogarithm formula

$$\mathrm{Li}_2(-z) + \mathrm{Li}_2\left(-\frac{1}{z}\right) = -\frac{\ln^2(z)}{2} + 2\mathrm{Li}_2(-1)$$

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \frac{1}{2}\mathrm{Li}_2(z^2)$$

Legendres chi function $\chi_v(z) = \frac{1}{2}[\mathrm{Li}_v(z) - \mathrm{Li}_v(-z)]$