ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^\infty \int_0^\infty \frac{\arctan(x^2)\arctan(y^4)}{x^2y^3}dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution by Shirvan Tahirov-Azerbaijan

$$\int_0^\infty \int_0^\infty \frac{\arctan(x^2)\arctan(y^4)}{x^2y^3}dxdy =$$

$$= \int_0^\infty \frac{\arctan(x^2)}{x^2}dx \,.\int_0^\infty \frac{\arctan(y^4)}{y^3}dy = X\,.Y$$

$$X = \int_0^\infty \frac{\arctan(x^2)}{x^2}dx \overset{IBP}{=} \left(-\frac{\arctan(x^2)}{x^2}\right)\Bigg|_0^\infty + 2\int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_0^\infty \frac{x^{-\frac{3}{4}}}{1+x}dx =$$

$$= \frac{1}{2}\beta\left(\frac{1}{4}\,;\,\frac{3}{4}\right) = \frac{\pi}{\sqrt{2}}$$

$$Y = \int_0^\infty \frac{\arctan(y^4)}{y^3}dy \overset{IBP}{=} \left(-\frac{\arctan(y^4)}{2y^2}\right)\Bigg|_0^\infty + 2\int_0^\infty \frac{y}{1+y^8}dy = \int_0^\infty \frac{dy}{1+y^4} =$$

$$= \frac{1}{4}\int_0^\infty \frac{y^{-\frac{3}{4}}}{1+y}dy = \frac{1}{4}\beta\left(\frac{1}{4}\,;\,\frac{3}{4}\right) = \frac{\pi}{2\sqrt{2}}$$

$$\int_0^\infty \frac{\arctan(x^2)}{x^2}dx \,.\, \int_0^\infty \frac{\arctan(y^4)}{y^3}dy = X\,.\,Y = \frac{\pi}{\sqrt{2}}\cdot\frac{\pi}{2\sqrt{2}} = \frac{\pi^2}{4}$$