ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_0^1\frac{\tanh^{-1}(xy)}{1-x^2y^2}dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\int_0^1\frac{\tanh^{-1}(xy)}{1-x^2y^2}dxdy=-\int_0^1\frac{\ln(x)\tanh^{-1}(x)}{1-x^2}dx=$$

$$=\frac{1}{2}\int_0^1\frac{\ln(x)\ln\left(\frac{1-x}{1+x}\right)}{1-x^2}dx=-\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{2k+1}\int_0^1 x^{2k+2n+1}\ln(x)\,dx=$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(2k+1)(2k+2n+2)^2}=\frac{1}{4}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(2k+1)(n+k+1)^2}$$

$$\left\{polygamma\ function:\psi^{(a)}(n+1)=\sum_{k=1}^{\infty}\frac{(-1)^{n+1}a!}{(k+n)^{a+1}};\ a\epsilon\{Z\geq 1\}\right\}$$

$$\Omega=\frac{1}{4}\sum_{k=0}^{\infty}\frac{\psi^{(1)}(k+1)}{2k+1}=\frac{1}{4}\sum_{k=0}^{\infty}\frac{\zeta(2)-H_k^{(2)}}{2k+1}=\frac{1}{4}\left(\zeta(2)\sum_{k=0}^{\infty}\frac{1}{2k+1}-\sum_{k=0}^{\infty}\frac{H_k^{(2)}}{2k+1}\right)=$$

$$=\frac{1}{4}\left(\zeta(2)\int_0^1\frac{1}{1-x^2}dx-\int_0^1\sum_{k=0}^{\infty}H_k^{(2)}x^{2k}\,dx\right)=\frac{1}{4}\left(\frac{1}{2}\zeta(2)[\ln\left(\frac{1-x}{1+x}\right)]_0^1-I\right)$$

$$\left\{we\ know\ <=>\sum_{n=0}^{\infty}x^nH_n^{(a)}=\frac{Li_a(x)}{1-x},\ \ Li_a(1)=\zeta(a)\right\}$$

$$I=\int_0^1\sum_{k=0}^{\infty}x^{2k}H_k^{(2)}dx=\int_0^1\frac{Li_2(x^2)}{1-x^2}dx,\ using\ IBP\ method$$

$$I=[\frac{1}{2}Li_2(x^2)]_0^1\int_0^1\frac{1}{1-x^2}dx-\int_0^1\frac{\ln(1-x^2)\ln\left(\frac{1-x}{1+x}\right)}{x}dx=$$

$$=\frac{\zeta(2)}{2}\int_0^1\frac{1}{1-x^2}dx-\int_0^1\frac{\ln(1-x^2)\ln\left(\frac{1-x}{1+x}\right)}{x}dx$$

$$\Omega=\frac{1}{4}\left(\frac{1}{2}\zeta(2)\int_0^1\frac{1}{1-x^2}dx-I\right)=\frac{1}{4}\int_0^1\frac{\ln(1-x^2)\ln\left(\frac{1-x}{1+x}\right)}{x}dx=$$

$$=\frac{1}{4}\int_0^1\frac{\ln^2(1-x)-\ln^2(1+x)}{x}dx=\frac{1}{4}\int_0^1\frac{\ln^2(x)}{1-x}dx-\frac{1}{4}\int_0^1\frac{\ln^2(1+x)}{x}dx=$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}\int_0^1 x^n \ln^2(x)dx-\frac{1}{4}J=\frac{1}{2}\zeta(3)-\frac{1}{4}J$$

$$J=\int_0^1\frac{\ln^2(1+x)}{x}dx, substitution\left\{\frac{1}{1+x}=y, \quad y\left[\frac{1}{2};\ 1\right]\right\}$$

$$J=\int_{\frac{1}{2}}^1\frac{\ln^2(y)}{y(1-y)}dy=\int_{\frac{1}{2}}^1\frac{\ln^2(y)}{y}dy+\int_{\frac{1}{2}}^1\frac{\ln^2(y)}{1-y}dy=$$

$$=\frac{\ln^3(2)}{3}+\int_0^1\frac{\ln^2(y)}{1-y}dy-\int_0^{\frac{1}{2}}\frac{\ln^2(y)}{1-y}dy=\frac{\ln^3(2)}{3}+2\zeta(3)-\frac{7}{4}\zeta(3)-\frac{\ln^3(2)}{3}=$$

$$=\frac{\zeta(3)}{4};\quad \Omega=\frac{1}{2}\zeta(3)-\frac{1}{4}J=\frac{1}{2}\zeta(3)-\frac{1}{16}\zeta(3)=\frac{7}{16}\zeta(3)$$

$$\int_0^1\int_0^1\frac{\tanh^{-1}(xy)}{1-x^2y^2}dxdy=\frac{7}{16}\zeta(3)$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\mathrm{I}=\int_0^1\int_0^1\frac{\tanh^{-1}(xy)}{1-x^2y^2}dx\,dy\Bigg]_{xy=m}=\int_0^1\frac{1}{x}\int_0^x\frac{\tanh^{-1}(m)}{1-m^2}dm\,dx\overset{IBP}{=}$$

$$\overset{IBP}{=}\log(x)\int_0^x\frac{\tanh^{-1}(m)}{1-m^2}dm\Bigg]_{x=0}^{x=1}-\int_0^1\frac{\log(x)\tanh^{-1}(x)}{1-x^2}dx=-\int_0^1\frac{\log(x)\tanh^{-1}(x)}{1-x^2}dx$$

$$\tanh^{-1}(x)=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\mathrm{I}=-\frac{1}{2}\int_0^1\frac{\log(x)\log\left(\frac{1+x}{1-x}\right)}{1-x^2}dx=\frac{1}{2}\int_0^1\frac{\log(x)\log\left(\frac{1-x}{1+x}\right)}{1-x^2}dx\Bigg]_{\frac{1-x}{1+x}=z}=$$

$$=\frac{1}{4}\int_0^1\frac{\log(z)\log\left(\frac{1-z}{1+z}\right)}{z}dz=$$

$$=\frac{1}{4}\int_0^1\frac{\log(z)\log(1-z)}{z}dz-\frac{1}{4}\int_0^1\frac{\log(z)\log(1+z)}{z}dz=$$

$$=\frac{1}{4}\zeta(3)-\frac{1}{4}\left\{-\frac{3}{4}\zeta(3)\right\}=\frac{7}{16}\zeta(3)$$