ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_0^1 \tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)dxdy$$

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Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\int_0^1 \tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)dxdy\,;IBP\to \quad u=\tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right),\ du=\frac{y(y^2-x^2)dx}{x^4+x^2y^2+y^4}$$

$$v=\int dx=x$$

$$\Omega=\int_0^1\left([\tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)x]_0^1-\int_0^1\frac{xy(y^2-x^2)}{x^4+x^2y^2+y^4}dx\right)dy=$$

$$\Omega=\int_0^1\tanh^{-1}\left(\frac{1}{y}+y\right)dy-\int_0^1\int_0^1\frac{xy(y^2-x^2)}{x^4+x^2y^2+y^4}dxdy=\Omega_1-\Omega_2$$

$$\Omega_1=\int_0^1\tanh^{-1}\left(y+\frac{1}{y}\right)dy\,,IBP\to \quad u=\tanh^{-1}\left(y+\frac{1}{y}\right),\ du=\frac{(1-y^2)}{1+y^2+y^4}dy$$

$$v=\int dy=y$$

$$\Omega_1=[\tanh^{-1}\left(y+\frac{1}{y}\right)y]_0^1-\int_0^1\frac{y(1-y^2)}{1+y^2+y^4}dy=\tanh^{-1}(2)-\int_0^1\frac{y-2y^3+y^5}{1-y^6}dy=$$

$$=\tanh^{-1}(2)-\sum_{n=0}^{\infty}\int_0^1 y^{6n+1}-2y^{6n+3}+y^{6n+5}dy=\frac{1}{2}\ln\left(\frac{1+2}{1-2}\right)-$$

$$-\sum_{n=0}^{\infty}\left(\frac{1}{6n+2}-\frac{2}{6n+4}+\frac{1}{6n+6}\right)=\frac{1}{2}\ln(3)-\frac{i\pi}{2}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(3n+1)(3n+2)}+$$

$$+\frac{1}{6}\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+2)}=\frac{1}{2}\ln(3)-\frac{i\pi}{2}-\frac{1}{6}\left(\psi^{(0)}\left(1-\frac{1}{3}\right)-\psi^{(0)}\left(\frac{1}{3}\right)\right)+$$

$$+\frac{1}{6}\left(\psi^{(0)}(1)-\psi^{(0)}\left(\frac{2}{3}\right)\right)=\frac{1}{2}\ln(3)-\frac{i\pi}{2}-\frac{\pi}{6\sqrt{3}}+\frac{\ln(3)}{4}-\frac{\pi}{12\sqrt{3}}=$$

$$=\frac{3\ln(3)}{4}-\frac{\pi}{4\sqrt{3}}-\frac{i\pi}{2}=-\frac{\pi\sqrt{3}}{12}+\frac{3\ln(3)}{4}-\frac{i\pi}{2}$$

$$\Omega_2=\int_0^1\int_0^1\frac{xy(y^2-x^2)}{x^4+x^2y^2+y^4}dxdy=$$

$$=\int_0^1\int_0^1\frac{xy^3}{x^4+x^2y^2+y^4}dxdy-\int_0^1\int_0^1\frac{x^3y}{x^4+x^2y^2+y^4}dxdy=0\ \{symmetry\}$$

$$\int_0^1\int_0^1\tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)dxdy=\Omega_1-\Omega_2=-\frac{\pi\sqrt{3}}{12}+\frac{3\ln(3)}{4}-\frac{i\pi}{2}$$

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Solution 2 by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1 \tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)dx\,dy\Bigg]_{\frac{y}{x}=m}=\int_0^1 x\int_0^{\frac{1}{x}}\tanh^{-1}\left(m+\frac{1}{m}\right)dm\,dx\overset{IBP}{=}$$

$$\overset{IBP}{=}\frac{x^2}{2}\int_0^{\frac{1}{x}}\tanh^{-1}\left(m+\frac{1}{m}\right)dm\Bigg|_{x=0}^{x=1}+\frac{1}{2}\int_0^1\tanh^{-1}\left(x+\frac{1}{x}\right)dx=\frac{1}{2}\int_0^1\tanh^{-1}\left(m+\frac{1}{m}\right)dm+$$

$$+\frac{1}{2}\int_0^1\tanh^{-1}\left(x+\frac{1}{x}\right)dx=\int_0^1\tanh^{-1}\left(x+\frac{1}{x}\right)dx$$

$$\tanh^{-1}\left(x+\frac{1}{x}\right)=z\Rightarrow\tanh\left(\tanh^{-1}\left(x+\frac{1}{x}\right)\right)=\tanh(z)\Rightarrow x+\frac{1}{x}=\tanh(z)\Rightarrow x+\frac{1}{x}$$

$$=\frac{e^{2z}-1}{e^{2z}+1}\Rightarrow e^{2z}=\frac{x^2+x+1}{x-1-x^2}\Rightarrow\log(e^{2z})=\log\left(\frac{x^2+x+1}{x-1-x^2}\right)\Rightarrow z$$

$$=\frac{1}{2}\log\left(\frac{x^2+x+1}{x-1-x^2}\right)$$

$$\tanh^{-1}\left(x+\frac{1}{x}\right)=\frac{1}{2}\log\left(\frac{x^2+x+1}{x-1-x^2}\right)$$

$$I=\frac{1}{2}\int_0^1\log\left(\frac{x^2+x+1}{x-1-x^2}\right)dx=\frac{1}{2}\left\{\int_0^1\log(x^2+x+1)\,dx-\int_0^1\log(x-1-x^2)\,dx\right\}$$

$$=\frac{1}{2}\left\{\left(\frac{3}{2}\log(3)-2+\frac{\pi}{2\sqrt{3}}\right)-\left(i\pi-2+\frac{\pi}{\sqrt{3}}\right)\right\}=-\frac{\pi\sqrt{3}}{12}+\frac{3}{4}\log(3)-\frac{i\pi}{2}$$