ROMANIAN MATHEMATICAL MAGAZINE

Prove the below closed form

$$I=\int_0^\infty \frac{J_{1/2}(x)J_{-1/2}(x)}{1+x^2}dx=\frac{e^2-1}{2e^2}$$

Where, $J_n(x)$ is the Bessel function of the first kind.

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$$I=\int_0^\infty \frac{J_{1/2}(x)J_{-1/2}(x)}{1+x^2}dx=\int_0^\infty \frac{\sqrt{\frac{2}{\pi x}}\sin(x)\sqrt{\frac{2}{\pi x}}\cos(x)}{1+x^2}dx=\frac{1}{\pi}\int_0^\infty \frac{\sin(2x)}{x(1+x^2)}dx$$

$$=\frac{1}{\pi}\int_0^\infty \frac{\sin(2x)}{x}dx-\frac{1}{\pi}\int_0^\infty \frac{x\sin(2x)}{1+x^2}dx=\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \frac{x\sin(2x)}{1+x^2}dx$$

$$\begin{cases} f(z)=\dfrac{2}{\pi}\displaystyle\int_0^\infty\left(\int_0^\infty f(t)\cos(xt)dt\right)\cos(zx)dx \\ f(z)=\dfrac{2}{\pi}\displaystyle\int_0^\infty\left(\int_0^\infty f(t)\sin(xt)dt\right)\sin(zx)dx \end{cases},\text{put } f(z)=e^{-z}$$

$$\Rightarrow\begin{cases} e^{-z}=\dfrac{2}{\pi}\displaystyle\int_0^\infty\left(\int_0^\infty e^{-t}\cos(xt)dt\right)\cos(zx)\,dx \\ e^{-z}=\dfrac{2}{\pi}\displaystyle\int_0^\infty\left(\int_0^\infty e^{-t}\sin(xt)dt\right)\sin(zx)\,dx \end{cases}$$

$$\Rightarrow\begin{cases} e^{-z}=\dfrac{2}{\pi}\displaystyle\int_0^\infty\frac{\cos(zx)}{1+x^2}dx \\ e^{-z}=\dfrac{2}{\pi}\displaystyle\int_0^\infty\frac{x\sin(zx)}{1+x^2}dx \end{cases}\Rightarrow\begin{cases} \dfrac{\pi}{2}e^{-z}=\displaystyle\int_0^\infty\frac{\cos(zx)}{1+x^2}dx \\ \dfrac{\pi}{2}e^{-z}=\displaystyle\int_0^\infty\frac{x\sin(zx)}{1+x^2}dx \end{cases}\Rightarrow\int_0^\infty\frac{x\sin(2x)}{1+x^2}dx$$

$$=\frac{\pi}{2}e^{-2}$$

$$I=\frac{1}{2}-\frac{1}{\pi}\left(\frac{\pi}{2}e^{-2}\right)=\frac{1}{2}-\frac{1}{2e^2}=\frac{e^2-1}{2e^2}$$