ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$I=\int_0^{\frac{\pi}{2}} \log\sqrt{1+\sin(x)+\cos(x)}\,dx = G-\frac{\pi}{8}\log(2)$$

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$$I=\int_0^{\frac{\pi}{2}} \log\sqrt{1+\sin(x)+\cos(x)}\,dx$$

$$=\int_0^{\frac{\pi}{2}} \log\sqrt{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}\,dx$$

$$=\int_0^{\frac{\pi}{2}} \log\sqrt{\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)^2+\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}\,dx$$

$$=\int_0^{\frac{\pi}{2}} \log\sqrt{2\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)\cos\left(\frac{x}{2}\right)}\,dx=\int_0^{\frac{\pi}{2}} \log\sqrt{2\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\,dx$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}} \log\left(2\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)dx$$

$$=\frac{1}{2}\left\{\int_0^{\frac{\pi}{2}} \log(2\sqrt{2})\,dx+\int_0^{\frac{\pi}{2}} \log\left(\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)dx+\int_0^{\frac{\pi}{2}} \log\left(\cos\left(\frac{x}{2}\right)\right)dx\right\}$$

$$=\frac{1}{2}\left\{\frac{3\pi}{4}\log(2)+\int_0^{\frac{\pi}{2}} \log\left(\sin\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)dx+\int_0^{\frac{\pi}{2}} \log\left(\cos\left(\frac{x}{2}\right)\right)dx\right\}$$

$$=\frac{1}{2}\left\{\frac{3\pi}{4}\log(2)+2\int_0^{\frac{\pi}{2}} \log\left(\cos\left(\frac{x}{2}\right)\right)dx\right\}=\frac{1}{2}\left\{\frac{3\pi}{4}\log(2)+4\int_0^{\frac{\pi}{4}} \log(\cos(x))\,dx\right\}$$

$$=\frac{1}{2}\left\{\frac{3\pi}{4}\log(2)+4\left\{\frac{1}{2}G-\frac{\pi}{4}\log(2)\right\}\right\}=G-\frac{\pi}{8}\log(2)$$

$$\text{Note: }\int_0^{\frac{\pi}{4}} \log(\cos(x))\,dx=\frac{1}{2}G-\frac{\pi}{4}\log(2)$$