ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log\left(\frac{\sin(x)}{\cos(y)}+\frac{\cos(x)}{\sin(y)}\right)dx\,dy=\frac{7}{8}\varsigma(3)+\frac{\pi^2}{4}\log(2)$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log\left(\frac{\sin(x)}{\cos(y)}+\frac{\cos(x)}{\sin(y)}\right)dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log\left(\frac{2\cos(x-y)}{\sin(2y)}\right)dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(2)\,dx\,dy+$$

$$+\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\cos(x-y))\,dx\,dy-\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\sin(2y))\,dx\,dy=\frac{\pi^2}{4}\log(2)+I_1-I_2$$

$$I_1=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\cos(x-y))\,dx\,dy\Bigg]_{x-y=m}=\int_0^{\frac{\pi}{2}}\int_{-y}^{\frac{\pi}{2}-y}\log(\cos(m))\,dm\,dy\overset{IBP}{=}$$

$$\overset{IBP}{=}y\int_{-y}^{\frac{\pi}{2}-y}\log(\cos(m))\,dm\Bigg]_{y=0}^{y=\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy-\int_0^{\frac{\pi}{2}}y\log(\cos(y))\,dy=$$

$$=\frac{\pi}{2}\int_{-\frac{\pi}{2}}^{0}\log(\cos(m))\,dm+\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy-\int_0^{\frac{\pi}{2}}y\log(\cos(y))\,dy=$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\log(\cos(m))\,dm+\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy-\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-y\right)\log(\sin(y))\,dy=$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\log(\cos(m))\,dm+\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy-\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\log(\sin(y))\,dy+\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy=$$

$$=2\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy=2\left\{-\frac{\pi^2}{8}\log(2)+\frac{7}{16}\varsigma(3)\right\}=-\frac{\pi^2}{4}\log(2)+\frac{7}{8}\varsigma(3)$$

$$I_2=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\sin(2y))\,dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(2)\,dx\,dy+\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\sin(y))\,dx\,dy+$$

$$+\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\cos(y))\,dx\,dy=\frac{\pi^2}{4}\log(2)+2\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\sin(y))\,dx\,dy=\frac{\pi^2}{4}\log(2)+$$

$$+\pi\int_0^{\frac{\pi}{2}}\log(\sin(y))\,dy=\frac{\pi^2}{4}\log(2)+\pi\left(-\frac{\pi}{2}\log(2)\right)=-\frac{\pi^2}{4}\log(2)$$

$$I=\frac{\pi^2}{4}\log(2)+I_1-I_2=\frac{7}{8}\varsigma(3)+\frac{\pi^2}{4}\log(2)$$

$$\text{Note: }\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy-\frac{\pi^2}{8}\log(2)+\frac{7}{16}\varsigma(3)$$