ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$I=\int_0^{\infty}\frac{\ln(x+1)}{(x+1)(x+2)^2(x+3)^3}dx$$

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$$I=\int_0^{\infty}\frac{\ln(x+1)}{(x+1)^2}dx-\frac{5}{4}\int_0^{\infty}\frac{\ln(x+1)}{(x+3)^2}dx+\frac{9}{4}\int_0^{\infty}\frac{x\ln(x+1)}{(x+1)(x+2)(x+3)}dx$$
$$+\frac{10}{4}\int_0^{\infty}\frac{\ln(x+1)}{(x+1)(x+2)(x+3)}dx=A+B+C+D+W$$

$$W+D=\frac{9}{4}\int_0^{\infty}\frac{\ln(x+1)\,x}{(x+1)(x+2)(x+3)}dx$$
$$+\frac{10}{4}\int_0^{\infty}\frac{\ln(x+1)}{(x+1)(x+2)(x+3)}dx\underbrace{=}_{x+2\to x}-2\int_2^{\infty}\frac{\ln(x-1)}{x(x^2-1)}dx$$
$$+\frac{9}{4}\int_2^{\infty}\frac{\ln(x-1)}{x^2-1}dx=$$

$$x\to\frac{1}{x}=-2\int_0^{\frac{1}{2}}\frac{x\ln(1-x)}{1-x^2}dx+\frac{9}{4}\int_0^{\frac{1}{2}}\frac{\ln(1-x)}{1-x^2}dx+\frac{9}{4}\int_0^{\frac{1}{2}}\frac{(x-1)\ln(x)}{1-x^2}dx$$
$$-\frac{1}{4}\int_0^{\frac{1}{2}}\frac{x\ln(x)}{1-x^2}dx=\Phi+\Upsilon+\Psi+\Lambda=W+D$$

$$\Phi=-2\int_0^{\frac{1}{2}}\frac{x\ln(1-x)}{1-x^2}dx\underbrace{=}_{IBP}-2\ln(2)\ln\left(\frac{3}{4}\right)+\int_0^{\frac{1}{2}}\frac{\ln(1-x^2)}{1-x}dx=-2\ln(2)\ln\left(\frac{3}{4}\right)$$
$$+\int_0^{\frac{1}{2}}\frac{\ln(1-x)}{1-x}dx+\int_0^{\frac{1}{2}}\frac{\ln(1+x)}{1-x}dx=-2\ln(2)\ln\left(\frac{3}{4}\right)-\frac{\ln(2)^2}{2}+P\to$$

$$P=\int_0^{1/2}\frac{\ln(1+x)}{1-x}dx\underbrace{=}_{1-x\to x}\int_{1/2}^{1}\frac{\ln(2-x)}{x}dx=\ln(2)\int_{\frac{1}{2}}^{1}\frac{\ln\left(1-\frac{x}{2}\right)}{x}dx$$
$$=(\ln(2))^2-\sum_{n=1}^{\infty}\frac{1}{n2^n}\int_{\frac{1}{2}}^{1}x^{n-1}dx=\ln(2))^2-\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n^2}+\sum_{n=1}^{\infty}\frac{\left(\frac{1}{4}\right)^n}{n^2}=$$

$$=(\ln 2)^2 - Li_2\left(\frac{1}{2}\right) + Li_2\left(\frac{1}{4}\right) \rightarrow \ \Phi = -\ln(2)\ln(3) + 3\ln(2)^2 + Li_2\left(\frac{1}{4}\right) - \frac{\pi^2}{12} \qquad note: Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{(ln2)^2}{2}$$

$$\Upsilon = \frac{9}{4}\int_0^{1/2} \frac{\ln(1-x)}{1-x^2}dx$$

$$= \frac{9}{8}\int_0^{1/2} \frac{\ln(1-x)}{1-x}dx + \frac{9}{8}\int_0^{1/2} \frac{\ln(1-x)}{1+x}dx = -\frac{9}{16}(ln2)^2 + \frac{9\ln(2)}{32}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)^n}{n+1} - \frac{9}{16}\sum_{n=0}^{\infty} \frac{\left(1-\left(\frac{1}{2}\right)^{n+1}\right)}{2^n(n+1)^2}$$

$$\rightarrow \quad \Upsilon = \frac{9}{8}\ln(2)\ln(3) + \frac{9}{4}(ln2)^2 - \frac{9\pi^2}{96} + \frac{9}{8}Li_2\left(\frac{1}{4}\right)$$

$$\Lambda = \frac{-1}{4}\int_0^{\frac{1}{2}} \frac{xln(x)}{1-x^2}dx$$

$$= \frac{-1}{4}\sum_{n=0}^{\infty}\int_0^{\frac{1}{2}} x^{2n+1}\ln(x)\,dx \underbrace{=}_{IBP} \frac{\ln(2)}{8}\sum_{n=0}^{\infty}\frac{1}{4^{n+1}(n+1)} + \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{2n+2}\int_0^{\frac{1}{2}} x^{2n+1}dx = \frac{1}{16}\sum_{n=0}^{\infty}\frac{2ln2(n+1)+1}{4^{n+1}(n+1)^2} =$$

$$\Lambda = \frac{-\ln(2)\ln(3)}{8} + \frac{(\ln 2)^2}{4} + \frac{1}{16}Li_2\left(\frac{1}{4}\right)$$

$$\Psi = \frac{9}{4}\int_0^{1/2} \frac{(x-1)\ln(x)}{1-x^2}dx$$

$$= \frac{-9}{4}\int_0^{1/2} \frac{\ln(x)}{x+1}dx \underbrace{=}_{IBP} \frac{9\ln(2)}{8}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})^n}{n+1} + \frac{9}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\int_0^{1/2} x^n dx = \frac{9\ln(2)}{8}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})^n}{n+1} + \frac{9}{8}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})^n}{(n+1)^2}$$

$$\Psi = \frac{9\ln(2)\ln(3)}{4} - \frac{9(\ln 2)^2}{4} - \frac{9}{4}Li_2\left(-\frac{1}{2}\right)$$

$$= \frac{9\ln(2)\ln(3)}{4} - \frac{27(ln2)^2}{8} - \frac{9}{8}Li_2\left(\frac{1}{4}\right) + \frac{9\pi^2}{48} \qquad note: Li_2(z) + Li_2(-z) = \frac{1}{2}Li_2(z^2)$$

$\rightarrow W+D=\Phi+\Upsilon+\Psi+\Lambda=\frac{17}{8}(ln2)^2+\frac{17}{8}Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{96}$

$$A=-\int_0^\infty \frac{\ln(x+1)}{(x+2)^2}dx \underbrace{=}_{IBP} \int_0^\infty \frac{dx}{(x+1)(x+2)}=\ln(\frac{x+1}{x+2}]_0^\infty=-\ln(2)$$

$$B=\frac{-5}{4}\int_0^\infty \frac{\ln(x+1)}{(x+3)^2}dx \underbrace{=}_{IBP} -\frac{5}{4}\int_0^\infty \frac{dx}{(x+1)(x+3)}=-\frac{5}{8}\ln(\frac{x+1}{x+3}]_0^\infty=\frac{5}{8}\ln(3)$$

$$C=\int_0^\infty \frac{\ln(x+1)}{(x+3)^3}dx \underbrace{=}_{IBP} -\frac{1}{4}\int_0^\infty \frac{1}{(x+1)(x+3)^2}dx$$
$$=\frac{-1}{16}\int_0^\infty (\frac{1}{x+1}-\frac{1}{x+3})dx+\frac{1}{8}\int_0^\infty \frac{1}{(x+3)^2}dx$$
$$=\frac{-1}{16}\ln(\frac{x+1}{x+3}]_0^\infty-\frac{1}{8(x+3)}]_0^\infty=-\frac{\ln(3)}{16}+\frac{1}{24}$$

ANSWER: I=A+B+C+D+W=$\frac{17}{16}Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{96}+\frac{17}{8}(\ln 2)^2-\frac{11}{16}\ln(3)-\ln(2)+\frac{1}{24}$