ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\Omega = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\sin(x+y+z)\ .\ \cos(x+y+z)}{\sqrt{x\sqrt{y\sqrt{z}}}}dxdydz = -\pi^{5/2}\frac{\sqrt[4]{15+11\sqrt{2}-2\sqrt{116+82\sqrt{2}}}}{4\sqrt[4]{8}\ .\ \Gamma\left(\frac{1}{4}\right).\Gamma\left(\frac{1}{8}\right)}$$

Proposed by Shirvan Tahirov-Azerbaijan, Ankush Kumar Parcha-India
Solution by Amin Hajiyev-Azerbaijan

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{\sin(x+y+z)\ .\ \cos(x+y+z)}{\sqrt{x\sqrt{y\sqrt{z}}}}dxdydz$$

$$= \frac{1}{2}\int_0^\infty \int_0^\infty \int_0^\infty \frac{\sin(2x+2y+2z)}{\sqrt{x\sqrt{y\sqrt{z}}}}dxdydz =$$

$$= \frac{1}{2}Im\left\{\int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{2xi}e^{2yi}e^{2zi}}{x^{\frac{1}{2}}.y^{\frac{1}{4}}.z^{\frac{1}{8}}}dxdydz =\right\}$$

$$= \frac{1}{2}Im\int_0^\infty \int_0^\infty \int_0^\infty e^{2ix}x^{-\frac{1}{2}}e^{2iy}y^{-\frac{1}{4}}e^{2iz}z^{-\frac{1}{6}}dxdydz =$$

$$= \frac{1}{2}Im(K\ .\ M\ .N)$$

$$K = \int_0^\infty e^{2xi}\,x^{-\frac{1}{2}}dx = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{2i^3}} = \frac{\sqrt{\pi}}{i\sqrt{2i}} = i^{-\frac{3}{2}}\sqrt{\frac{\pi}{2}} = e^{-\frac{3}{4}\ln(i^2)}\sqrt{\frac{\pi}{2}} = -e^{-\frac{3}{4}i\pi}\sqrt{\frac{\pi}{2}}$$

$$M = \int_0^\infty e^{2yi}y^{-\frac{1}{4}}dy = \frac{\Gamma\left(\frac{3}{4}\right)}{(-2i)^{\frac{3}{4}}} = \frac{\Gamma\left(\frac{3}{4}\right)i^{\frac{3}{4}}}{\sqrt[4]{8}} = \frac{\Gamma\left(\frac{3}{4}\right)e^{\frac{3}{8}\ln(i^2)}}{\sqrt[4]{8}} = e^{\frac{3}{8}i\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt[4]{8}}$$

$$N = \int_0^\infty e^{2zi}z^{-\frac{1}{8}}dz = \frac{\Gamma\left(\frac{7}{8}\right)}{(-2i)^{\frac{7}{8}}} = \frac{\Gamma\left(\frac{7}{8}\right)i^{\frac{7}{8}}}{2^{\frac{7}{8}}} = e^{\frac{7}{16}i\pi}\frac{\Gamma\left(\frac{7}{8}\right)}{2^{\frac{7}{8}}}$$

$$\Omega = \frac{1}{2}Im(K\ .\ M\ .N) = -\frac{1}{2}Im\left\{\sqrt{\frac{\pi}{2}}\ .e^{\frac{7}{16}i\pi}\frac{\Gamma\left(\frac{7}{8}\right)}{2^{\frac{7}{8}}}\ .e^{-\frac{3}{4}i\pi}.e^{\frac{3}{8}i\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt[4]{8}}\right\} =$$

$$-\frac{1}{2}Im\left\{\sqrt{\frac{\pi}{2}}.\ \frac{\Gamma\left(\frac{7}{8}\right)\Gamma\left(\frac{3}{4}\right)}{2^{\frac{7}{8}}.\ \sqrt[4]{8}}\ .e^{\frac{1}{16}i\pi}\right\} = -\frac{1}{2}.\ \sqrt{\frac{\pi}{2}}.\frac{\Gamma\left(\frac{7}{8}\right)\Gamma\left(\frac{3}{4}\right)}{2^{\frac{7}{8}}.\ \sqrt[4]{8}}\ .\sin\left(\frac{\pi}{16}\right)$$

$$\Omega = -\frac{1}{2} \cdot \sqrt{\frac{\pi}{2}} \frac{\pi^2 \sin\left(\frac{\pi}{16}\right)}{2^{\frac{7}{8}} \cdot \sqrt[4]{8} \cdot \sin\left(\frac{\pi}{4}\right) \cdot \sin\left(\frac{\pi}{8}\right) \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{8}\right)}$$

$$= -\pi^{5/2} \frac{\sqrt[4]{15 + 11\sqrt{2} - 2\sqrt{116 + 82\sqrt{2}}}}{4\sqrt[4]{8} \cdot \Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{1}{8}\right)}$$