ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_R \left(\frac{1}{x}-\frac{\sin(x)}{x^2}\right)^2 dx=\frac{\pi}{3}$$

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$$\Omega=\int_{-\infty}^{\infty}\left(\frac{x-\sin(x)}{x^2}\right)^2 dx=2\int_0^{\infty}\frac{x^2-2x\sin(x)+sin^2(x)}{x^4}dx=$$

$$=2\int_0^{\infty}L^{-1}\left\{\frac{1}{x^4}\right\}(s)L\{x^2-2x\sin(x)+sin^2(x)\}(s)ds$$

$$Laplace\ transform: L\{f\}(s)=\int_0^{\infty}f(t)e^{-st}dt$$

$$Inverse\ Laplace\ transform: L^{-1}\{F(s)\}(t)=f(t)=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT}e^{st}F(s)ds$$

$$L\{t^n\}(s)=\frac{n!}{s^{n+1}};\ L^{-1}\left\{\frac{a}{s^m}\right\}(t)=\frac{a\,t^{m-1}}{(m-1)!}\ \rightarrow\ L^{-1}\left\{\frac{1}{x^4}\right\}(s)=\frac{s^3}{3!}=\frac{s^3}{6}$$

$$L\{x^2\}(s)=\frac{2!}{s^{2+1}}=\frac{2}{s^3}$$

$$L\{t^n\sin(at)\}(s)=(-1)^n\frac{d^n}{ds^n}\left(\frac{a}{s^2+a^2}\right)\ \rightarrow\ L\{2x\sin(x)\}(s)=\frac{4s}{(s^2+1)^2}$$

$$L\{\cos(at)\}(s)=\frac{s}{s^2+a^2}\ \rightarrow\ L\{sin^2(x)\}(s)=L\left\{\frac{1}{2}-\frac{\cos(2x)}{2}\right\}(s)=$$

$$=\frac{1}{2s}-\frac{s}{2(s^2+4)}=\frac{2}{s(s^2+4)}$$

$$\Omega=2\int_0^{\infty}\frac{s^3}{6}\left(\frac{2}{s^3}-\frac{4s}{(s^2+1)^2}+\frac{2}{s(s^2+4)}\right)ds=$$

$$=\frac{8}{3}\int_0^{\infty}\frac{1}{s^2+1}ds-\frac{8}{3}\int_0^{\infty}\frac{1}{s^2+4}ds-\frac{4}{3}\int_0^{\infty}\frac{1}{(s^2+1)^2}ds=$$

$$=\frac{8}{3}\cdot\frac{\pi}{2}-\frac{8}{3}\cdot\frac{\pi}{4}-\frac{4}{3}\cdot\frac{\pi}{4}=\frac{\pi}{3}$$