ROMANIAN MATHEMATICAL MAGAZINE

Prove the below closed form:

$$\int_0^{\frac{\pi}{4}} \frac{\sin(x)+\cos(x)+\tan(x)}{(1+\sin(x))(1+\cos(x))(1+\tan(x))}dx = \ln\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)-2\sqrt{2}+3$$

Proposed by Ankush Kumar Parcha-India

Solution by Pham Duc Nam-Vietnam

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{\sin(x)+\cos(x)+\tan(x)}{(1+\sin(x))(1+\cos(x))(1+\tan(x))}dx = ?$$

* By letting: $t=\tan(x)$, we have: $\Omega = \int_0^1 \left(-\frac{t}{\sqrt{1+t^2}}+\frac{1}{1+\sqrt{1+t^2}}+\frac{t}{1+t}\right)dt$

$$= \left(-\sqrt{1+t^2}+t-\ln(1+t)\right)\Big|_0^1 + \underbrace{\int_0^1 \frac{1}{1+\sqrt{1+t^2}}dt}_{\sqrt{1+t^2}=u+t} =$$

$$= 2-\sqrt{2}-\ln(2)+\int_{\sqrt{2}-1}^1 \frac{u^2+1}{u(1+u)^2}du$$

$$= 2-\sqrt{2}-\ln(2)+\int_{\sqrt{2}-1}^1 \left(\frac{1}{u}-\frac{2}{(1+u)^2}\right)du = 2-\sqrt{2}-\ln(2)+\left(\frac{2}{1+u}+\ln(u)\right)\Big|_{\sqrt{2}-1}^1$$

$$= 2-\sqrt{2}-\ln(2)-\sqrt{2}+1-\ln\left(\sqrt{2}-1\right) = 3-2\sqrt{2}-\ln\left(2\sqrt{2}-2\right) =$$

$$= 3-2\sqrt{2}+\ln\left(\frac{1}{2\sqrt{2}-2}\right) = \ln\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)-2\sqrt{2}+3$$