ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_{R^+}\frac{\log^2(x)\sin^3(x)}{x}dx=\frac{\pi\gamma^2}{4}-\frac{\pi\gamma}{4}\log(3)+\frac{\pi^3}{48}-\frac{\pi}{8}\log^2(3)$$

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$$\Omega=\int_0^\infty\frac{\log^2(x)}{x}\sin^3(x)\,dx=\lim_{a\to 0}\frac{d^2}{da^2}\int_0^\infty x^{a-1}\sin^3(x)dx=$$

$$=\lim_{a\to 0}\frac{d^2}{da^2}\int_0^\infty L_x^{-1}\{x^{a-1}\}(s)L_x\{\sin^3(x)\}(s)ds=$$

$$=\lim_{a\to 0}\frac{d^2}{da^2}\frac{6}{\Gamma(1-a)}\int_0^\infty\frac{s^{-a}}{(1+s^2)(9+s^2)}ds=$$

$$=\frac{3}{4}\lim_{a\to 0}\frac{d^2}{da^2}\frac{1}{\Gamma(1-a)}\left(\int_0^\infty\frac{s^{-a}}{s^2+1}ds-\int_0^\infty\frac{s^{-a}}{s^2+9}ds\right)$$

$$I(a)=\int_0^\infty\frac{s^{-a}}{1+s^2}ds,\qquad\{s^2=t,\quad dt=2\sqrt{t}ds,\quad t[0;\infty]\}$$

$$I(a)=\frac{1}{2}\int_0^\infty\frac{t^{\frac{1}{2}-\frac{a}{2}-1}}{(1+t)^{\frac{1}{2}-\frac{a}{2}+\frac{a}{2}+\frac{1}{2}}}dt=\frac{1}{2}\beta\left(\frac{1}{2}-\frac{a}{2};\frac{1}{2}+\frac{a}{2}\right)$$

$$J(a)=\int_0^\infty\frac{s^{-a}}{9+s^2}ds=\frac{1}{9}\int_0^\infty\frac{s^{-a}}{\frac{s^2}{9}+1}ds\quad\left\{\frac{s^2}{9}=t,\quad dt=\frac{2}{3}\sqrt{t}ds,\quad t[0;\infty]\right\}$$

$$J(a)=\frac{3^{-a}}{6}\int_0^\infty\frac{t^{\frac{1}{2}-\frac{a}{2}-1}}{(1+t)^{\frac{1}{2}-\frac{a}{2}+\frac{1}{2}+\frac{a}{2}}}dt=\frac{3^{-a-1}}{2}\beta\left(\frac{1}{2}-\frac{a}{2};\frac{1}{2}+\frac{a}{2}\right)$$

$$\Omega=\frac{3}{8}\lim_{a\to 0}\frac{d^2}{da^2}\frac{(1-3^{-a-1})\beta\left(\frac{1}{2}-\frac{a}{2};\frac{1}{2}+\frac{a}{2}\right)}{\Gamma(1-a)}=\frac{3}{8}\lim_{a\to 0}\frac{d^2}{da^2}(1-3^{-1-a})\frac{\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)}{\Gamma(1-a)}$$

$$note:\left\{\Gamma(1-\alpha)\Gamma(1+\alpha)=\pi\alpha\csc(\pi\alpha),\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)=\pi\sec\left(\frac{\pi\alpha}{2}\right)\right\}$$

$$\Omega=\frac{3\pi}{8}\lim_{a\to 0}\frac{d^2}{da^2}\frac{\left(1-3^{-1-a}\right)\sec\left(\frac{\pi a}{2}\right)}{\Gamma(1-a)}=\frac{3\pi}{8}(\Omega_1-\Omega_2)$$

$$\Omega_1=\lim_{a\to 0}\left(\frac{\sec\left(\frac{\pi a}{2}\right)}{\Gamma(1-a)}\right)^{(2)}$$

$$=\lim_{a\to 0}\frac{\sec\left(\frac{\pi a}{2}\right)}{4\Gamma(1-a)}\left(4\psi^{(0)}(1-a)^2-4\psi^{(1)}(1-a)+\pi^2\left(\tan^2\left(\frac{\pi a}{2}\right)+\sec^2\left(\frac{\pi a}{2}\right)\right)\right.$$
$$\left.+4\pi\tan\left(\frac{\pi a}{2}\right)\psi^{(0)}(1-a)\right)=$$

$$=\psi^{(0)}(1)^2-\psi^{(1)}(1)+\frac{\pi^2}{4}=\gamma^2-\frac{\pi^2}{6}+\frac{\pi^2}{4}=\gamma^2+\frac{\pi^2}{12}$$

$$\Omega_2=\frac{1}{3}\lim_{a\to 0}\left(\frac{3^{-a}\sec\left(\frac{\pi a}{2}\right)}{\Gamma(1-a)}\right)^{(2)}$$

$$=\frac{1}{3}\lim_{a\to 0}\frac{3^{-a}\sec^3\left(\frac{\pi a}{2}\right)}{4\Gamma(1-a)}\left(\pi^2 sin^2\left(\frac{\pi a}{2}\right)+\pi^2+2\pi\sin(\pi a)\left(\psi^{(0)}(1-a)-\log(3)\right)\right.$$
$$\left.+4\cos^2\left(\frac{\pi a}{2}\right)\left(\psi^{(0)}(1-a)^2-\psi^{(1)}(1-a)-2\log(3)\,\psi^{(0)}(1-a)+\log^2(3)\right)\right)=$$

$$=\frac{1}{12}\left(\pi^2+4\psi^{(0)}(1)^2-4\psi^{(1)}(1)-8\log(3)\,\psi^{(0)}(1)+4\log^2(3)\right)=$$

$$=\frac{\pi^2}{12}+\frac{\gamma^2}{3}-\frac{\pi^2}{18}+\frac{2\gamma\log(3)}{3}+\frac{\log^2(3)}{3}=\frac{\pi^2}{36}+\frac{\gamma^2}{3}+\frac{2\gamma\log(3)}{3}+\frac{\log^2(3)}{3}$$

$$\Omega=\frac{3\pi}{8}(\Omega_1-\Omega_2)=\frac{3\pi}{8}\left(\frac{\pi^2}{18}+\frac{2\gamma^2}{3}-\frac{2\gamma\log(3)}{3}-\frac{\log^2(3)}{3}\right)=$$

$$=\frac{\pi^3}{48}+\frac{\pi\gamma^2}{4}-\frac{\gamma\,\pi\log(3)}{4}-\frac{\pi\log^2(3)}{8}$$