ROMANIAN MATHEMATICAL MAGAZINE

Prove the below closed form

$$\Omega=\int_0^1\int_0^1\int_0^1 \log\left(\frac{1}{x^y y^z z^x}\right)\frac{dxdydz}{1+xyz}=\frac{9}{4}\zeta(3)-\frac{1}{4}\zeta(2)+3\log\left(\frac{4}{e}\right)$$

Where, ζ(3) is the Apery's constant.

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Solution by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^1\int_0^1\int_0^1 \log\left(\frac{1}{x^y y^z z^x}\right)\frac{dxdydz}{1+xyz}=-\int_0^1\int_0^1\int_0^1 \log(x^y y^z z^x)\frac{dxdydz}{1+xyz}=$$

$$=-3\int_0^1\int_0^1\int_0^1 \log(x^y)\frac{dxdydz}{1+xyz}=-3\int_0^1\int_0^1\int_0^1 y\log(x)\frac{dxdydz}{1+xyz}=$$

$$=-3\sum_{k=0}^{\infty}(-1)^k\int_0^1\int_0^1\int_0^1 y^{k+1}z^k x^k\log(x)\,dx\,dy\,dz=$$

$$=3\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}\int_0^1\int_0^1 y^{k+1}z^k\,dy\,dz=3\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2(k+2)}\int_0^1 z^k dz=$$

$$=3\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^3(k+2)}=3\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3(k+1)}=$$

$$=3\sum_{k=1}^{\infty}(-1)^{k+1}\left[\frac{1}{k^3}-\frac{1}{k^2}+\frac{1}{k}-\frac{1}{k+1}\right]=$$

$$=3\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k+1}\right\}=$$

$$=3\left\{\frac{3}{4}\zeta(3)-\frac{1}{2}\zeta(2)+2\log(2)-1\right\}=\frac{9}{4}\zeta(3)-\frac{1}{4}\zeta(2)+3\log\left(\frac{4}{e}\right)$$