ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1\int_0^1\int_0^1 \frac{(1+xyz)(3+xyz)}{(2+xyz)(4+xyz)}dxdydz$$

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$$\textit{Useful identity}: \int_0^1\int_0^1\int_0^1 f(xyz)dxdydz = \frac{1}{2}\int_0^1 f(x)ln^2(x)dx$$

$$\Omega = \int_0^1\int_0^1\int_0^1 \frac{(1+xyz)(3+xyz)}{(2+xyz)(4+xyz)}dxdydz = \frac{1}{2}\int_0^1 \frac{(1+x)(3+x)}{(2+x)(4+x)}ln^2(x)dx =$$

$$\frac{1}{2}\underbrace{\int_0^1 ln^2(x)dx}_{2} - \frac{1}{4}\int_0^1 \frac{ln^2(x)}{2+x}dx - \frac{3}{4}\int_0^1 \frac{ln^2(x)}{4+x}dx$$

$$\int_0^1 \frac{ln^n(x)}{a+x}dx = -(-1)^n n!\, Li_{n+1}\left(-\frac{1}{a}\right)$$

$$\Omega = 2 - \frac{1}{4}\left(-Li_3\left(-\frac{1}{2}\right)\right) - \frac{3}{4}\left(-2Li_3\left(-\frac{1}{4}\right)\right)$$

$$\int_0^1\int_0^1\int_0^1 \frac{(1+xyz)(3+xyz)}{(2+xyz)(4+xyz)}dxdydz = 2 + \frac{1}{2}Li_3\left(-\frac{1}{2}\right) + \frac{3}{2}Li_3\left(-\frac{1}{4}\right)$$