ROMANIAN MATHEMATICAL MAGAZINE

Prove the below closed form

$$I=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\left(\frac{\sqrt{xy}+\sqrt{yz}}{\sqrt{yz}+\sqrt{zx}}\right)dx\,dy\,dz=\frac{48}{5}\log(2)-\frac{33}{10}$$

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Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\left(\frac{\sqrt{xy}+\sqrt{yz}}{\sqrt{yz}+\sqrt{zx}}\right)dx\,dy\,dz=3\int_0^1\int_0^1\int_0^1\frac{\sqrt{y}(\sqrt{x}+\sqrt{z})}{\sqrt{z}(\sqrt{x}+\sqrt{y})}dx\,dy\,dz$$

Let $\sqrt{x}\to m,\ \sqrt{y}\to n,\ \sqrt{z}\to p$

$$I=24\int_0^1\int_0^1\int_0^1\frac{mn^2(m+p)}{m+n}dp\,dn\,dm=24\int_0^1\int_0^1\frac{m\left(m+\frac{1}{2}\right)n^2}{m+n}dn\,dm$$

$$=24\int_0^1 m^3\left(m+\frac{1}{2}\right)\log(1+m)\,dm+12\int_0^1 m\left(m+\frac{1}{2}\right)dm$$

$$-24\int_0^1 m^2\left(m+\frac{1}{2}\right)dm-24\int_0^1 m^3\left(m+\frac{1}{2}\right)\log(m)\,dm$$

$$=24\left(\frac{320\log(2)-67}{800}\right)+12\left(\frac{7}{12}\right)-24\left(\frac{5}{12}\right)-24\left(-\frac{57}{800}\right)$$

$$=\frac{48}{5}\log(2)-\frac{33}{10}$$