ROMANIAN MATHEMATICAL MAGAZINE

Prove the below closed form

$$\Omega=\int_0^{\frac{\pi}{2}}\log\left(1+\frac{\sin(x)}{\sin(2x)}\right)dx=\frac{4G}{3}$$

where, *G* is the *Catalan's constant*.

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^{\frac{\pi}{2}}\log\left(1+\frac{\sin(x)}{\sin(2x)}\right)dx=\int_0^{\frac{\pi}{2}}\log\left(\frac{\sin(2x)+\sin(x)}{\sin(2x)}\right)dx=\int_0^{\frac{\pi}{2}}\log\left(\frac{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)}{\sin(2x)}\right)dx$$

$$=\int_0^{\frac{\pi}{2}}\log(2)\,dx+\int_0^{\frac{\pi}{2}}\log\left(\sin\left(\frac{3x}{2}\right)\right)dx+\int_0^{\frac{\pi}{2}}\log\left(\cos\left(\frac{x}{2}\right)\right)dx-\int_0^{\frac{\pi}{2}}\log(\sin(2x))\,dx$$

$$=\frac{\pi}{2}\log(2)+\frac{2}{3}\int_0^{\frac{3\pi}{4}}\log(\sin(x))\,dx+2\int_0^{\frac{\pi}{4}}\log(\cos(x))\,dx-\frac{1}{2}\int_0^{\pi}\log(\sin(x))\,dx$$

$$=\frac{\pi}{2}\log(2)+\frac{2}{3}\left\{-\frac{3\pi}{4}\log(2)+\frac{1}{2}G\right\}+2\left\{\frac{1}{2}G-\frac{\pi}{4}\log(2)\right\}-\frac{1}{2}\{-\pi\log(2)\}=\frac{4G}{3}$$

$$\text{Note: }\int_0^{\frac{3\pi}{4}}\log(\sin(x))\,dx=\int_0^{\frac{3\pi}{4}}\left\{-\log(2)-\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k}\right\}dx$$

$$=-\frac{3\pi}{4}\log(2)-\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\frac{3\pi}{4}}\cos(2kx)\,dx=-\frac{3\pi}{4}\log(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{3\pi}{2}k\right)}{k^2}$$

$$=-\frac{3\pi}{4}\log(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^2}=-\frac{3\pi}{4}\log(2)+\frac{1}{2}G$$