ROMANIAN MATHEMATICAL MAGAZINE

Prove the below closed form

$$\Omega = \int_0^1 \int_0^1 \frac{\log(1 - x^4 y^4)}{xy} dxdy = -\frac{\zeta(3)}{16}$$

where $\zeta(3)$ is the Apery's constant.

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We know that:

$$\int_0^1 \int_0^1 f(xy)\, dxdy = -\int_0^1 \log(x)\, f(x)\, dx$$

Then we can write:

$$\Omega = \int_0^1 \int_0^1 \frac{\log(1 - x^4 y^4)}{xy} dxdy = -\int_0^1 \frac{\log(x) \log(1 - x^4)}{x} dx \Bigg]_{x^4 \to x} =$$

$$= -\frac{1}{16} \int_0^1 \frac{\log(x) \log(1 - x)}{x} dx = -\frac{1}{16} \int_0^1 \frac{\log(x)}{x} \left(-\sum_{k=1}^{\infty} \frac{x^k}{k} \right) dx =$$

$$= \frac{1}{16} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 x^{k-1} \log(x)\, dx = -\frac{1}{16} \sum_{k=1}^{\infty} \frac{1}{k^3} = -\frac{\zeta(3)}{16}$$