ROMANIAN MATHEMATICAL MAGAZINE

Prove the below closed form

$$\Omega=\int_0^1\int_0^1 \tan^{-1}\left(\frac{1-xy}{1+xy}\right)\cot^{-1}\left(\frac{1-xy}{1+xy}\right)\frac{dxdy}{\log(xy)}=-G+\frac{\pi}{4}\log(2)$$

where G is the Catalan's constant.

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We know that:

$$\int_0^1\int_0^1 f(xy)\,dxdy=-\int_0^1 \log(x)\,f(x)\,dx$$

Then we can write

$$\Omega=\int_0^1\int_0^1 \tan^{-1}\left(\frac{1-xy}{1+xy}\right)\cot^{-1}\left(\frac{1-xy}{1+xy}\right)\frac{dxdy}{\log(xy)}=-\int_0^1 \tan^{-1}\left(\frac{1-x}{1+x}\right)\cot^{-1}\left(\frac{1-x}{1+x}\right)dx$$

$$=-\int_0^1 \tan^{-1}\left(\frac{1-x}{1+x}\right)\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1-x}{1+x}\right)\right)dx$$

$$=-\int_0^1\left(\frac{\pi}{4}-\tan^{-1}(x)\right)\left(\frac{\pi}{4}+\tan^{-1}(x)\right)dx=-\int_0^1\left(\frac{\pi^2}{16}-\arctan^2(x)\right)dx$$

$$=-\frac{\pi^2}{16}+\int_0^1 \arctan^2(x)\,dx=-\frac{\pi^2}{16}+\left(\frac{\pi^2}{16}+\frac{\pi}{4}\log(2)-G\right)=\frac{\pi}{4}\log(2)-G$$

$$\text{Note: }\int_0^1 \arctan^2(x)\,dx=\frac{\pi^2}{16}+\frac{\pi}{4}\log(2)-G$$