ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^{\infty} \frac{\frac{\frac{x}{x^2+1+x^2}}{x^4+x^2+1+x^2+x^4}}{\frac{x^2}{\frac{x}{\pi}}}\,dx = \frac{2-\sqrt{1+\sqrt{2}}}{2\sqrt{2}}$$

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$\boldsymbol{After\ simplification}$:

$$I = \frac{1}{\pi}\int_0^{\infty} \frac{1}{(1+2x^2)(1+2x^2+2x^4)}dx =$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{dx}{(1+2x^2)} - \frac{1}{\pi}\int_0^{\infty} \frac{(1+2x^2)}{(1+2x^2+2x^4)}dx = \frac{1}{\sqrt{2}} - \frac{1}{\pi}\int_0^{\infty} \frac{\frac{1}{x^2}+2}{2x^2+\frac{1}{x^2}+2}dx =$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\pi}\int_0^{\infty} \frac{\frac{1}{x^2}+2}{\left(\sqrt{2}x-\frac{1}{x}\right)^2+2+2\sqrt{2}}dx \underset{x\to\frac{1}{\sqrt{2}x}}{\overset{\omega}{=}} \frac{1}{\sqrt{2}} - \frac{1}{\pi}\int_0^{\infty} \frac{\sqrt{2}\left(\frac{1}{x^2}+1\right)}{\left(\sqrt{2}x-\frac{1}{x}\right)^2+2+2\sqrt{2}}dx =$$

$$I = \frac{1}{\sqrt{2}} - \frac{1}{2\pi}\int_0^{\infty} \frac{\frac{1}{x^2}+2+\sqrt{2}\left(\frac{1}{x^2}+1\right)}{\left(\sqrt{2}x-\frac{1}{x}\right)^2+2+2\sqrt{2}}dx = \frac{1}{\sqrt{2}} - \frac{1+\sqrt{2}}{2\pi}\int_0^{\infty} \frac{\sqrt{2}+\frac{1}{x^2}}{\left(\sqrt{2}x-\frac{1}{x}\right)^2+2+2\sqrt{2}}dx$$

$$Now\ \ substitute\ \sqrt{2}x - \frac{1}{x} = u$$

$$I = \frac{1}{\sqrt{2}} - \frac{1+\sqrt{2}}{2\pi}\int_{-\infty}^{\infty} \frac{du}{u^2+2+2\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1+\sqrt{2}}{2\pi}\cdot\frac{\pi}{\sqrt{2+2\sqrt{2}}} = \frac{1}{\sqrt{2}} - \frac{\sqrt{1+\sqrt{2}}}{2\sqrt{2}}$$

$$Therefore,\qquad I = \frac{2-\sqrt{1+\sqrt{2}}}{2\sqrt{2}}$$