ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$I = \iiint_{\{[0,1]^3\}} \frac{(1+xyz)(3+xyz)}{(2+xyz)(4+xyz)} dxdydz$$

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Solution by Pratham Prasad-India

By, partial fractions

$$I = \iiint_{\{[0,1]^3\}} \left(-\frac{3}{8}\frac{1}{1+\frac{xyz}{4}} - \frac{1}{4}\frac{1}{1+\frac{xyz}{2}} + 1\right) dxdydz$$

$$I = \iiint_{\{[0,1]^3\}} \left(-\frac{3}{8}\sum_{r=0}^{\infty}\left(-\frac{xyz}{4}\right)^r - \frac{1}{4}\sum_{r=0}^{\infty}\left(-\frac{xyz}{2}\right)^r + 1\right) dxdydz$$

$$I = -\frac{3}{8}\sum_{r=0}^{\infty}\iiint_{\{[0,1]^3\}}\left(-\frac{xyz}{4}\right)^r - \frac{1}{4}\sum_{r=0}^{\infty}\iiint_{\{[0,1]^3\}}\left(-\frac{xyz}{2}\right)^r + 1\, dxdydz$$

$$I = \frac{3}{2}\sum_{r=0}^{\infty}\frac{\left(\frac{-1}{4}\right)^{r+1}}{(r+1)^3} + \frac{1}{2}\sum_{r=0}^{\infty}\frac{\left(\frac{-1}{2}\right)^{r+1}}{(r+1)^3} + 1, \qquad I = \frac{3}{2}Li_3\left(-\frac{1}{4}\right) + \frac{1}{2}Li_3\left(-\frac{1}{2}\right) + 1$$

$$I = \frac{1}{2}\left(3Li_3\left(-\frac{1}{4}\right) + Li_3\left(-\frac{1}{2}\right)\right) + 1$$