ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^{\frac{\pi}{2}} x\ ln\left(1+\frac{1}{csc(x)}+\frac{1}{sec(x)}\right)dx$$

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Solution by Pratham Prasad-India

$$\int_0^{\frac{\pi}{2}} x\ln\left(1+\frac{1}{\csc(x)}+\frac{1}{\sec(x)}\right)dx=\int_0^{\frac{\pi}{2}} x\ln(1+\sin(x)+\cos(x))dx=$$

$$x=2u$$

$$=4\int_0^{\frac{\pi}{4}} u\ \ln(1+\sin(2u)+\cos(2u))\,du=4\int_0^{\frac{\pi}{4}} u\ \ln\left(2cos^2(u)+2\cos(u)\sin(u)\right)du=$$

$$=4\int_0^{\frac{\pi}{4}} u\ \ln(2cos(u)(\cos(\mathrm{u})+\sin(u))\,du=4\int_0^{\frac{\pi}{4}} u\ \ln\left(2\sqrt{2}cos(u)\cos(\frac{\pi}{4}-u)\right)du=$$

$$=4\int_0^{\frac{\pi}{4}} u\ \ln(2\sqrt{2})\,du+4\int_0^{\frac{\pi}{4}} u\ \ln(\cos(u))\,du+\ 4\int_0^{\frac{\pi}{4}} u\ \ln\left(\cos(\frac{\pi}{4}-u)\right)du=$$

$$replace\ (\frac{\pi}{4}-u)\ by\ u\ in\ the\ last\ integral\ by\ u$$

$$=4\int_0^{\frac{\pi}{4}} u\ \ln(2\sqrt{2})\,du+4\int_0^{\frac{\pi}{4}} u\ \ln(\cos(u))\,du+\ 4\int_0^{\frac{\pi}{4}} (\frac{\pi}{4}-u)\ \ln(\cos(u))\,du=$$

$$=4\int_0^{\frac{\pi}{4}} u\ \ln(2\sqrt{2})\,du+4\int_0^{\frac{\pi}{4}} u\ \ln(\cos(u))\,du+\pi\int_0^{\frac{\pi}{4}} \ln(\cos(u))\,du-4\int_0^{\frac{\pi}{4}} u\ \ln(\cos(u))\,du=$$

$$=\frac{12}{2}\int_0^{\frac{\pi}{4}} u\ \ln(2)\,du+\pi\int_0^{\frac{\pi}{4}} \ln(\cos(u))\,du=\frac{3\pi^2}{16}\ln(2)+\pi(\frac{G}{2}-\frac{\pi}{4}\ln(2))=\frac{G\pi}{2}-\frac{\pi^2}{16}\ln(2)$$