ROMANIAN MATHEMATICAL MAGAZINE

Prove the below closed form

$$I=\int_0^1\int_0^1\int_0^1 \frac{\ln(x+y+z)}{x+y+z}\,dx\,dy\,dz = 9\ln(2)-\frac{27}{4}\ln(3)-3\ln^2(2)+\frac{9}{4}\ln^2(3)$$

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$$\text{Let } x+y+z\to m$$

$$I=\int_0^1\int_0^1\int_{x+y}^{x+y+1}\frac{\ln(m)}{m}\,dm\,dy\,dx \overset{IBP}{=} \int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}\,dm\,dx-\int_0^1\int_0^1\frac{y\ln(x+y+1)}{x+y+1}\,dy\,dx\Bigg]_{x+y+1=m}$$

$$+\int_0^1\int_0^1\frac{y\ln(x+y)}{x+y}\,dy\,dx\Bigg]_{x+y=m}$$

$$=\int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}\,dm\,dx-\int_0^1\int_{x+1}^{x+2}\frac{(m-1-x)\ln(m)}{m}\,dm\,dx$$

$$+\int_0^1\int_x^{x+1}\frac{(m-x)\ln(m)}{m}\,dm\,dx$$

$$=\int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}\,dm\,dx-\int_0^1\int_{x+1}^{x+2}\ln(m)\,dm\,dx+\int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}\,dm\,dx+\int_0^1 x\int_{x+1}^{x+2}\frac{\ln(m)}{m}\,dm\,dx$$

$$+\int_0^1\int_x^{x+1}\ln(m)\,dm\,dx-\int_0^1 x\int_x^{x+1}\frac{\ln(m)}{m}\,dm\,dx$$

$$=2\int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}\,dm\,dx-\int_0^1\int_{x+1}^{x+2}\ln(m)\,dm\,dx+\int_0^1 x\int_{x+1}^{x+2}\frac{\ln(m)}{m}\,dm\,dx+\int_0^1\int_x^{x+1}\ln(m)\,dm\,dx$$

$$-\int_0^1 x\int_x^{x+1}\frac{\ln(m)}{m}\,dm\,dx$$

$$I_1=\int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}\,dm\,dx \overset{IBP}{=} \int_2^3\frac{\ln(m)}{m}\,dm-\int_0^1\frac{x\ln(x+2)}{x+2}\,dx+\int_0^1\frac{x\ln(x+1)}{x+1}\,dx$$

$$=\frac{3}{2}\ln^2(3)-2\ln^2(2)-3\ln(3)+4\ln(2)$$

$$I_2=\int_0^1\int_{x+1}^{x+2}\ln(m)\,dm\,dx \overset{IBP}{=} \int_2^3\ln(m)\,dm-\int_0^1 x\ln(x+2)\,dx+\int_0^1 x\ln(x+1)\,dx$$

$$=\frac{9}{2}\ln(3)-4\ln(2)-\frac{3}{2}$$

$$I_3=\int_0^1 x\int_{x+1}^{x+2}\frac{\ln(m)}{m}\,dm\,dx \overset{IBP}{=} \frac{1}{2}\int_2^3\frac{\ln(m)}{m}\,dm-\frac{1}{2}\int_0^1\frac{x^2\ln(x+2)}{x+2}\,dx+\frac{1}{2}\int_0^1\frac{x^2\ln(x+1)}{x+1}\,dx$$

$$=-\frac{3}{4}\ln^2(3)+\ln^2(2)+\frac{15}{4}\ln(3)-4\ln(2)-\frac{3}{4}$$

$$I_4 = \int_0^1 \int_x^{x+1} \ln(m)\, dm\, dx \overset{IBP}{=} \int_1^2 \ln(m)\, dm - \int_0^1 x\ln(x+1)\, dx + \int_0^1 x\ln(x)\, dx = 2\ln(2) - \frac{3}{2}$$

$$I_5 = \int_0^1 x \int_x^{x+1} \frac{\ln(m)}{m}\, dm\, dx \overset{IBP}{=} \frac{1}{2}\int_1^2 \frac{\ln(m)}{m}\, dm - \frac{1}{2}\int_0^1 \frac{x^2\ln(x+1)}{x+1}\, dx + \frac{1}{2}\int_0^1 \frac{x^2\ln(x)}{x}\, dx$$

$$= \ln(2) - \frac{3}{4}$$

$$I = 2I_1 - I_2 + I_3 + I_4 - I_5 = 9\ln(2) - \frac{27}{4}\ln(3) - 3\ln^2(2) + \frac{9}{4}\ln^2(3)$$