ROMANIAN MATHEMATICAL MAGAZINE

## Prove the below closed form

$$I=\int_0^1\int_0^1\int_0^1\frac{1}{(1+x)(1+yx)(1+xyz)}dx\,dy\,dz=\frac{7}{8}\varsigma(3)-\frac{\pi^2}{12}\log(2)+\frac{1}{6}\log^3(2)$$

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$$I=\int_0^1\int_0^1\int_0^1\frac{1}{(1+x)(1+yx)(1+xyz)}dz\,dy\,dx=\int_0^1\int_0^1\frac{\log(1+xy)}{xy(1+x)(1+yx)}dy\,dx$$

$$\text{Let } xyz\to m$$

$$I=\int_0^1\frac{1}{x(1+x)}\int_0^x\frac{\log(1+m)}{m(1+m)}dm\,dx\overset{IBP}{=}\left[(\log(x)-\log(1+x))\int_0^x\frac{\log(1+m)}{m(1+m)}dm\right]_0^1$$

$$-\int_0^1\frac{\log(x)\log(1+x)-\log^2(1+x)}{x(1+x)}dx$$

$$=-\log(2)\int_0^1\frac{\log(1+m)}{m(1+m)}dm-\int_0^1\frac{\log(x)\log(1+x)}{x(1+x)}dx+\int_0^1\frac{\log^2(1+x)}{x(1+x)}dx$$

$$I_1=\int_0^1\frac{\log(1+m)}{m(1+m)}dm=\int_0^1\frac{\log(1+m)}{m}dm-\int_0^1\frac{\log(1+m)}{1+m}dm=\frac{\pi^2}{12}-\frac{1}{2}\log^2(2)$$

$$I_2=\int_0^1\frac{\log(x)\log(1+x)}{x(1+x)}dx=\int_0^1\frac{\log(x)\log(1+x)}{x}dx-\int_0^1\frac{\log(x)\log(1+x)}{1+x}dx$$

$$=-\frac{3}{4}\varsigma(3)+\frac{1}{8}\varsigma(3)=-\frac{5}{8}\varsigma(3)$$

$$I_3=\int_0^1\frac{\log^2(1+x)}{x(1+x)}dx=\int_0^1\frac{\log^2(1+x)}{x}dx-\int_0^1\frac{\log^2(1+x)}{1+x}dx=\frac{1}{4}\varsigma(3)-\frac{1}{3}\log^3(2)$$

$$I=-\log(2)\,I_1-I_2+I_3=\frac{7}{8}\varsigma(3)-\frac{\pi^2}{12}\log(2)+\frac{1}{6}\log^3(2)$$