ROMANIAN MATHEMATICAL MAGAZINE

**Prove the below closed form:**

$$I=\int_0^1\int_0^1\left(\frac{x+y}{x-y}\right)\tan^{-1}\left(\frac{x-y}{x+y}\right)dx\,dy=2G-\frac{\pi}{4}\log(2)-\frac{\log(2)}{2}$$

**Where, G is a Catalan's constant.**

*Proposed by Ankush Kumar Parcha-India*

*Solution by Togrul Ehmedov-Azerbaijan*

$$I=\int_0^1\int_0^1\left(\frac{x+y}{x-y}\right)\tan^{-1}\left(\frac{x-y}{x+y}\right)dx\,dy=\int_0^1\int_0^1\left(\frac{1+y/x}{1-y/x}\right)\tan^{-1}\left(\frac{1-y/x}{1+y/x}\right)dx\,dy$$

*Let $y/x\to m$*

$$I=\int_0^1 x\int_0^{\frac{1}{x}}\left(\frac{1+m}{1-m}\right)\tan^{-1}\left(\frac{1-m}{1+m}\right)dm\,dx\overset{IBP}{=}$$

$$=\left[\frac{x^2}{2}\int_0^{\frac{1}{x}}\left(\frac{1+m}{1-m}\right)\tan^{-1}\left(\frac{1-m}{1+m}\right)dm\right]_0^1+\frac{1}{2}\int_0^1\left(\frac{1+x}{x-1}\right)\tan^{-1}\left(\frac{x-1}{1+x}\right)dx=$$

$$=\frac{1}{2}\int_0^1\left(\frac{1+m}{1-m}\right)\tan^{-1}\left(\frac{1-m}{1+m}\right)dm\Bigg]_{m\to x}+\frac{1}{2}\int_0^1\left(\frac{1+x}{x-1}\right)\tan^{-1}\left(\frac{x-1}{1+x}\right)dx=$$

$$=\frac{1}{2}\int_0^1\left(\frac{1+x}{1-x}\right)\tan^{-1}\left(\frac{1-x}{1+x}\right)dx+\frac{1}{2}\int_0^1\left(\frac{1+x}{x-1}\right)\tan^{-1}\left(\frac{x-1}{1+x}\right)dx=$$

$$=\int_0^1\left(\frac{1+x}{1-x}\right)\tan^{-1}\left(\frac{1-x}{1+x}\right)dx=\frac{\pi}{4}\int_0^1\left(\frac{1+x}{1-x}\right)dx-\int_0^1\left(\frac{1+x}{1-x}\right)\tan^{-1}(x)\,dx=$$

$$=\frac{\pi}{4}\int_0^1\left(\frac{1+x}{1-x}\right)dx-2\int_0^1\frac{\tan^{-1}(x)}{1-x}dx+\int_0^1\tan^{-1}(x)\,dx$$

*Use the integration by parts formula*

ROMANIAN MATHEMATICAL MAGAZINE

$$I = \frac{\pi}{4}\left\{-2\log(1-x)\right]_0^1 - 1\Big\} - 2\left\{-\log(1-x)\tan^{-1}(x)\right]_0^1 + \int_0^1 \frac{\log(1-x)}{1+x^2}dx\Bigg\} + \int_0^1 \tan^{-1}(x)\,dx$$

$$= \log(1-x)\left\{-\frac{\pi}{2} + 2\tan^{-1}(x)\right\}\Bigg]_0^1 - \frac{\pi}{4} - 2\int_0^1 \frac{\log(1-x)}{1+x^2}dx + \int_0^1 \tan^{-1}(x)\,dx$$

$$= -\frac{\pi}{4} - 2\int_0^1 \frac{\log(1-x)}{1+x^2}dx + \int_0^1 \tan^{-1}(x)\,dx$$

$$= -\frac{\pi}{4} - 2\left\{\frac{\pi}{8}\log(2) - G\right\} + \left\{-\frac{\log(2)}{2} + \frac{\pi}{4}\right\} = 2G - \frac{\pi}{4}\log(2) - \frac{\log(2)}{2}$$