ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$I=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\sqrt{\frac{\sqrt{x}}{\sqrt{y}+\sqrt{z}}}\,dx\,dy\,dz=\frac{128}{35}\left(2-\sqrt{2}\right)$$

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Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\int_0^1\left\{\sqrt{\frac{\sqrt{x}}{\sqrt{y}+\sqrt{z}}}+\sqrt{\frac{\sqrt{y}}{\sqrt{x}+\sqrt{z}}}+\sqrt{\frac{\sqrt{z}}{\sqrt{x}+\sqrt{y}}}\right\}dx\,dy\,dz$$

$$=\int_0^1\int_0^1\int_0^1\sqrt{\frac{\sqrt{x}}{\sqrt{y}+\sqrt{z}}}\,dx\,dy\,dz+\int_0^1\int_0^1\int_0^1\sqrt{\frac{\sqrt{y}}{\sqrt{x}+\sqrt{z}}}\,dx\,dy\,dz$$

$$+\int_0^1\int_0^1\int_0^1\sqrt{\frac{\sqrt{z}}{\sqrt{x}+\sqrt{y}}}\,dx\,dy\,dz=3\int_0^1\int_0^1\int_0^1\sqrt{\frac{\sqrt{x}}{\sqrt{y}+\sqrt{z}}}\,dx\,dy\,dz$$

$$=\frac{12}{5}\int_0^1\int_0^1\frac{1}{\sqrt{\sqrt{y}+\sqrt{z}}}\,dy\,dz$$

$$\text{Let }\sqrt{\sqrt{y}+\sqrt{z}}=t$$

$$I=\frac{12}{5}\int_0^1\int_{\sqrt{\sqrt{z}}}^{\sqrt{1+\sqrt{z}}}\left(4t^2-4\sqrt{z}\right)dt\,dz=\frac{48}{5}\int_0^1\int_{\sqrt{\sqrt{z}}}^{\sqrt{1+\sqrt{z}}}t^2\,dt\,dz-\frac{48}{5}\int_0^1\int_{\sqrt{\sqrt{z}}}^{\sqrt{1+\sqrt{z}}}\sqrt{z}\,dt\,dz$$

$$=\frac{48}{5}\left\{\frac{16\sqrt{2}-4}{35}\right\}-\frac{48}{5}\left\{\frac{88\sqrt{2}-92}{105}\right\}=\frac{128}{35}\left(2-\sqrt{2}\right)$$