ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$I=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\frac{\log(x+y)}{\sqrt{x+y}}\,dx\,dy\,dz=8\sqrt{2}\log(2)-64\left(\frac{\sqrt{2}-1}{3}\right)$$

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Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\int_0^1\left\{\frac{\log(x+y)}{\sqrt{x+y}}+\frac{\log(x+z)}{\sqrt{x+z}}+\frac{\log(z+y)}{\sqrt{z+y}}\right\}dx\,dy\,dz=$$

$$=\int_0^1\int_0^1\int_0^1\frac{\log(x+y)}{\sqrt{x+y}}\,dx\,dy\,dz+\int_0^1\int_0^1\int_0^1\frac{\log(x+z)}{\sqrt{x+z}}\,dx\,dy\,dz=$$

$$+\int_0^1\int_0^1\int_0^1\frac{\log(z+y)}{\sqrt{z+y}}\,dx\,dy\,dz=3\int_0^1\int_0^1\int_0^1\frac{\log(x+y)}{\sqrt{x+y}}\,dx\,dy\,dz=$$

$$=3\int_0^1\int_0^1\int_0^1\frac{\log(x+y)}{\sqrt{x+y}}\,dy\,dx$$

Let $\sqrt{x+y}=m$

$$I=12\int_0^1\int_{\sqrt{x}}^{\sqrt{x+1}}\log(m)\,dm\,dx=$$

$$=12\int_0^1\{\sqrt{x+1}\log(\sqrt{x+1})-\sqrt{x+1}-\sqrt{x}\log(\sqrt{x})+\sqrt{x}\}dx=$$

$$=12\left\{\frac{2\sqrt{2}}{3}\log(2)-\frac{16\sqrt{2}}{9}+\frac{16}{9}\right\}=8\sqrt{2}\log(2)-64\left(\frac{\sqrt{2}-1}{3}\right)$$