ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_0^1 \arctan(x^2+y^2-2xy)\,dxdy$$

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$$\Omega=\int_0^1\int_0^1 \arctan\left((y-x)^2\right)dxdy=\int_0^1\int_{y-1}^{y} \arctan(x^2)\,dxdy$$

Applying Integration by parts on the outer Integral and replacing y with x after that,

$$=\int_0^1 \arctan(x^2)\,dx-\int_0^1 x\left(\arctan(x^2)-\arctan\left((x-1)^2\right)\right)dx$$

$$=\int_0^1 \arctan(x^2)\,dx-\int_0^1 x\left(\arctan(x^2)\right)dx+\int_0^1 x\left(\arctan\left((1-x)^2\right)\right)dx$$

$$1-x=t \text{ and replace } t \text{ with } x \text{ in the last integral, to get:}$$

$$=\int_0^1 \arctan(x^2)\,dx-\int_0^1 x\left(\arctan(x^2)\right)dx+\int_0^1 (1-x)\left(\arctan(x^2)\right)dx$$

$$=2\left(\int_0^1 \arctan(x^2)\,dx-\int_0^1 x\left(\arctan(x^2)\right)dx\right)$$

$$x^2=t \text{ and replace } t \text{ with } x \text{ in the last integral, to get:}$$

$$=2\left(\int_0^1 \arctan(x^2)\,dx\right)-\int_0^1 \arctan(x)\,dx=2(A)-B$$

$$I=2(A)-B,\qquad A=\int_0^1 \arctan(x^2)\,dx$$

Applying Integration by parts,

$$A=\frac{\pi}{4}-\int_0^1\frac{2x^2}{1+x^4}dx,\qquad A=\frac{\pi}{4}-\int_0^1\frac{(x^2-1)+(x^2+1)}{1+x^4}dx$$

$$A=\frac{\pi}{4}-\int_0^1\frac{(x^2-1)}{1+x^4}dx-\int_0^1\frac{(x^2+1)}{1+x^4}dx$$

$$A=\frac{\pi}{4}-\int_0^1\frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-2}dx-\int_0^1\frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+2}dx$$

$$\text{substitute } x+\frac{1}{x}=t \text{ in the first integral and } x-\frac{1}{x}=t \text{ in the second integral,}$$

$$A=\frac{\pi}{4}+\int_{2}^{\infty}\frac{1}{t^2-2}dt-\int_{-\infty}^{0}\frac{1}{t^2+2}dt,\qquad A=\frac{\pi}{4}+\frac{\ln(1+\sqrt{2})}{\sqrt{2}}-\frac{\pi}{2\sqrt{2}}$$

$$B=\int_0^1\arctan(x)\,dx$$

Applying Integration by parts,

$$B=\frac{\pi}{4}-\int_0^1\frac{x}{1+x^2}dx,\quad B=\frac{\pi}{4}-\frac{1}{2}\int_0^1\frac{2x}{1+x^2}dx$$

$x^2=t$ *and replace* t *with* x *in the last integral, to get:*

$$B=\frac{\pi}{4}-\frac{1}{2}\int_0^1\frac{1}{1+x}dx,\quad B=\frac{\pi}{4}-\frac{1}{2}\ln(2)$$

Now,

$$I=2(A)-B$$

$$I=2\left(\frac{\pi}{4}+\frac{\ln(1+\sqrt{2})}{\sqrt{2}}-\frac{\pi}{2\sqrt{2}}\right)-\frac{\pi}{4}+\frac{1}{2}\ln(2),$$

$$I=\frac{\pi}{2}+\sqrt{2}\ln(1+\sqrt{2})-\frac{\pi}{\sqrt{2}}-\frac{\pi}{4}+\frac{1}{2}\ln(2)$$

$$I=\frac{\pi}{4}+\sqrt{2}\ln(1+\sqrt{2})-\frac{\pi}{\sqrt{2}}+\frac{1}{2}\ln(2),\quad I=\frac{1}{2}\ln(2)+\sqrt{2}\ln(1+\sqrt{2})+\frac{\pi}{4}(1-\sqrt{8})$$

$$\int_0^1\int_0^1\arctan(x^2+y^2-2xy)\,dxdy=\frac{1}{2}\ln(2)+\sqrt{2}\ln(1+\sqrt{2})+\frac{\pi}{4}(1-\sqrt{8})$$