ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$I=\int_0^\infty\int_0^\infty \frac{e^{-x}x^4\log(x)\left(\log(y)-\log(x)\right)}{(x^2+y^2)^2}dxdy=\frac{\pi}{4}(\gamma-1)$$

Where, γ is Euler-Mascheroni constant

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Solution by Togrul Ehmedov-Azerbaijan

Let y/x=t

$$I=\int_0^\infty\int_0^\infty \frac{e^{-x}x^4\log(x)\left(\log(y)-\log(x)\right)}{(x^2+y^2)^2}dxdy=\int_0^\infty x\log(x)e^{-x}\,dx\int_0^\infty \frac{\log(t)}{(1+t^2)^2}dt$$

$$=I_1*I_2$$

$$I_1=\int_0^\infty x\log(x)e^{-x}\,dx \overset{IBP}{=} -xe^{-x}\log(x)\Big|_0^\infty+\int_0^\infty e^{-x}\log(x)\,dx+\int_0^\infty e^{-x}\,dx$$

$$=\int_0^\infty e^{-x}\log(x)\,dx+1=-\gamma+1$$

$$I_2=\int_0^\infty \frac{\log(t)}{(1+t^2)^2}dt=\int_0^1 \frac{\log(t)}{(1+t^2)^2}dt+\int_1^\infty \frac{\log(t)}{(1+t^2)^2}dt\Bigg]_{t\to1/t}$$

$$=\int_0^1 \frac{\log(t)}{(1+t^2)^2}dt-\int_0^1 \frac{t^2\log(t)}{(1+t^2)^2}dt$$

$$=\int_0^1 \frac{(1-t^2)\log(t)}{(1+t^2)^2}dt \overset{IBP}{=} \frac{t}{1+t^2}\log(t)\Big|_0^1-\int_0^1\frac{dt}{1+t^2}=-\int_0^1\frac{dt}{1+t^2}=-\frac{\pi}{4}$$

$$I=I_1*I_2=\frac{\pi}{4}(\gamma-1)$$