ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$I=\iint_0^1 \tan^{-1}\left(\frac{x+y}{1-xy}\right)\ln(xy)\,dxdy$$

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$$I=\iint_0^1 \tan^{-1}\left(\frac{x+y}{1-xy}\right)\ln(xy)\,dxdy$$

$$=\iint_0^1 \tan^{-1}(x)\ln(xy)\,dxdy+\iint_0^1 \tan^{-1}(y)\ln(xy)\,dxdy$$

$$A=\iint_0^1 \tan^{-1}(x)\ln(xy)\,dxdy\ .\ \ B=\iint_0^1 \tan^{-1}(y)\ln(xy)\,dxdy$$

$$A=\iint_0^1 \tan^{-1}(x)\ln(xy)\,dxdy=\frac{\partial}{\partial a}\Big|_{a=0}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\int_0^1 x^{2n+a-1}dx\int_0^1 y^a dy$$

$$\frac{\partial}{\partial a}\Big|_{a=0}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)(2n+a)(a+1)}=\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{4(2n-1)n^2}=B=A$$

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{2(2n-1)n^2}=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)n^2}$$

$$=\frac{-\pi^2}{24}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{2n-1}-\frac{2(-1)^n}{n}-\frac{(-1)^n}{n^2}$$

$$Answer=\frac{-\pi^2}{24}+4\left(\frac{-\pi}{4}\right)+2\ln(2)+\frac{\pi^2}{12}=\frac{\pi^2}{24}-\pi+\ln(4)$$