ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 \frac{\ln^2 x}{(x+1)(x^2+x\varphi^2+\varphi)}dx = \frac{\varphi\pi^2}{6} - 2\varphi^2 Li_3\left(-\frac{1}{\varphi}\right) - \frac{3}{2}\varphi^2\zeta(3)$$

$$\varphi = golden\ ratio, \zeta(3) = Apery's\ Constant\ ,$$

$$Li_3(z) = trilogarithmic\ function$$

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$$use,\quad \varphi^2 = 1+\varphi\ \ and\ \ 1+\varphi^{-1} = \varphi\ \ and\ \ \frac{1}{\varphi-1} = \varphi$$

$$x^2+x\varphi^2+\varphi = x^2+x(1+\varphi)+\varphi = (x+1)(x+\varphi)$$

$$Now,\quad I = \int_0^1 \frac{\ln^2 x}{(x+1)(x^2+x\varphi^2+\varphi)}dx = \int_0^1 \frac{\ln^2 x}{(x+1)^2(x+\varphi)}dx$$

$$= \int_0^1 \ln^2 x\left(\frac{\frac{1}{(\varphi-1)}}{(x+1)^2} - \frac{\frac{1}{(\varphi-1)^2}}{x+1} + \frac{\frac{1}{(\varphi-1)^2}}{x+\varphi}\right)dx$$

$$= \phi\int_0^1 \frac{\ln^2 x}{(1+x)^2}dx - \varphi^2\int_0^1 \frac{\ln^2 x}{x+1}dx + \varphi^2\int_0^1 \frac{\ln^2 x}{x+\varphi}dx = \phi A - \varphi^2 B + \varphi^2 C$$

$$Now,$$

$$A = \int_0^1 \frac{\ln^2 x}{(1+x)^2}dx = \int_0^1 \ln^2 x\left(1-2x+3x^2-4x^3+\cdots\right)dx = \Gamma 3\left(\frac{1}{1^3}-\frac{2}{2^3}+\frac{3}{3^3}-\frac{4}{4^3}+\cdots\right)$$
$$= 2\eta(2) = \zeta(2)$$

$$Now, \int_0^1 \frac{\ln^2 x}{x-a}dx = -\frac{1}{a}\sum_{n=1}^{\infty}\int_0^1 (\ln^2 x)\left(\frac{x}{a}\right)^{n-1}dx = -\frac{1}{a}\sum_{n=1}^{\infty}\frac{\Gamma 3}{a^{n-1}n^3} = -2\sum_{n=1}^{\infty}\frac{\left(\frac{1}{a}\right)^n}{n^3} = -2Li_3\left(\frac{1}{a}\right)$$

$$put\ \ a=-1\ \ \Rightarrow B = \int_0^1 \frac{\ln^2 x}{x+1}dx = -2Li_3(-1) = 2\eta(3) = \frac{3}{2}\zeta(3)$$

$$put\ \ a=-\varphi\ \Rightarrow C=\int_0^1 \frac{ln^2x}{x+\varphi}dx=-2Li_3\left(-\frac{1}{\varphi}\right)$$

$$Therefore,\qquad I=\phi\zeta(2)-\varphi^2\frac{3}{2}\zeta(3)+-2Li_3\left(-\frac{1}{\varphi}\right)\varphi^2=\frac{\varphi\pi^2}{6}-2\varphi^2Li_3\left(-\frac{1}{\varphi}\right)-\frac{3}{2}\varphi^2\zeta(3)$$