ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 \frac{\ln(1+x)}{x(2x+1)(3x+1)}dx = -5Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{4}-\frac{29}{2}ln^2 2+8(ln2)(ln3)$$

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$$I=\int_0^1 \frac{\ln(1+x)}{x(2x+1)(3x+1)}dx \underbrace{=}_{x\to x-1} \int_1^2 \frac{lnx}{(x-1)(2x-1)(3x-2)}dx \underbrace{=}_{x\to\frac{1}{x}} \int_1^{\frac{1}{2}} \frac{xlnx}{(1-x)(2-x)(3-2x)}dx$$

$$=\int_1^{\frac{1}{2}}(lnx)\left(\frac{1}{1-x}+\frac{2}{2-x}-\frac{6}{3-2x}\right)dx=\int_1^{\frac{1}{2}}(lnx)\left\{(1-x)^{-1}+\left(1-\frac{x}{2}\right)^{-1}-2\left(1-\frac{2x}{3}\right)^{-1}\right\}dx$$

$$=F(1)+F\left(\frac{1}{2}\right)-2F\left(\frac{2}{3}\right)$$

$$here,\ \ F(a)=\int_1^{\frac{1}{2}}(lnx)\,(1-ax)^{-1}dx=\sum_{n=1}^{\infty}a^{n-1}\int_1^{\frac{1}{2}}(lnx)\,x^{n-1}dx$$

$$Now,\ \ \int_1^{\frac{1}{2}}(lnx)\,x^{n-1}dx=\left[(lnx)\frac{x^n}{n}-\frac{x^n}{n^2}\right]_1^{\frac{1}{2}}=-\frac{ln2}{n2^n}+\frac{1}{n^2}-\frac{1}{n^2 2^n}$$

$$Therefore,\ \ \ F(a)=\frac{1}{a}(ln2)\left(1-\frac{a}{2}\right)+\frac{1}{a}Li_2(a)-\frac{1}{a}Li_2\left(\frac{a}{2}\right)$$

$$Note:\ \ Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{ln^2 2}{2}\Rightarrow F(1)=\frac{\pi^2}{12}-\frac{ln^2 2}{2}$$

$$\Rightarrow F\left(\frac{1}{2}\right)=2(ln2)(ln3)-5ln^2 2+\frac{\pi^2}{6}-2Li_2\left(\frac{1}{4}\right)$$

$$\Rightarrow F\left(\frac{2}{3}\right)=\frac{3}{2}ln^2 2-\frac{3}{2}(ln2)(ln3)+\frac{3}{2}Li_2\left(\frac{2}{3}\right)-\frac{3}{2}Li_2\left(\frac{1}{3}\right)\Rightarrow I=F(1)+F\left(\frac{1}{2}\right)-2F\left(\frac{2}{3}\right)$$

$$=\frac{\pi^2}{12}-\frac{ln^2 2}{2}+2(ln2)(ln3)-5ln^2 2+\frac{\pi^2}{6}-2Li_2\left(\frac{1}{4}\right)-3ln^2 2+3(ln2)(ln3)-3Li_2\left(\frac{2}{3}\right)+3Li_2\left(\frac{1}{3}\right)$$

$$\Rightarrow I=\frac{\pi^2}{4}-\frac{17ln^2 2}{2}+5(ln2)(ln3)-2Li_2\left(\frac{1}{4}\right)-3\left(Li_2\left(\frac{2}{3}\right)-Li_2\left(\frac{1}{3}\right)\right)$$

Now use the following identities:

$$Li_2(z) + Li_2(1-z) = \frac{\pi^2}{6} - ln(x)ln(1-x) \underbrace{\Rightarrow}_{z=\frac{1}{3}} Li_2\left(\frac{2}{3}\right) + Li_2\left(\frac{1}{3}\right)$$

$$= \frac{\pi^2}{6} - ln^2 3 + (ln2)(ln3) \quad \dots\dots\dots\dots(A)$$

$$Li_2(1-z) + Li_2(1-z^{-1}) = -\frac{ln^2 z}{2} \underbrace{\Rightarrow}_{z=\frac{2}{3}} Li_2\left(\frac{1}{3}\right) + Li_2\left(-\frac{1}{2}\right) = (ln2)(ln3) - \frac{ln^2 2}{2} - \frac{ln^2 3}{2} \quad \dots\dots\dots(B)$$

$$Li_2(z) + Li_2(-z) = \frac{1}{2}Li_2(z^2) \underbrace{\Rightarrow}_{z=\frac{1}{2}} Li_2\left(\frac{1}{2}\right) + Li_2\left(-\frac{1}{2}\right) = \frac{1}{2}Li_2\left(\frac{1}{4}\right) \quad \dots\dots\dots(C)$$

Now use, $(A) - 2(B) + 2(C)$

$$\Rightarrow Li_2\left(\frac{2}{3}\right) - Li_2\left(\frac{1}{3}\right) + 2Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{6} - ln^2 3 + (ln2)(ln3) - 2(ln2)(ln3) + ln^2 2 + ln^2 3 + Li_2\left(\frac{1}{4}\right)$$

$$\Rightarrow Li_2\left(\frac{2}{3}\right) - Li_2\left(\frac{1}{3}\right) + \frac{\pi^2}{6} - ln^2 2 = \frac{\pi^2}{6} - (ln2)(ln3) + ln^2 2 + Li_2\left(\frac{1}{4}\right)$$

$$\Rightarrow Li_2\left(\frac{2}{3}\right) - Li_2\left(\frac{1}{3}\right) = 2ln^2 2 - (ln2)(ln3) + Li_2\left(\frac{1}{4}\right)$$

$$\Rightarrow I = \frac{\pi^2}{4} - \frac{17ln^2 2}{2} + 5(ln2)(ln3) - 2Li_2\left(\frac{1}{4}\right) - 3\left(2ln^2 2 - (ln2)(ln3) + Li_2\left(\frac{1}{4}\right)\right)$$

$$\Rightarrow I = \frac{\pi^2}{4} - \frac{29ln^2 2}{2} + 8(ln2)(ln3) - 5Li_2\left(\frac{1}{4}\right)$$