ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\iint_{[0,1]^2} \tan^{-1}\left(\frac{x+y}{x+1}\right) dxdy = \frac{1}{8}\left(3\pi - 22\ln 2 + 7\ln 5 - 2\cot^{-1}2\right)$$

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$$I = \int_{x=0}^{1}\int_{y=0}^{1} \tan^{-1}\left(\frac{x+y}{x+1}\right) dydx \underbrace{=}_{\substack{x\to x-1\\ y\to 1-y}} \int_{x=1}^{2}\int_{y=0}^{1} \tan^{-1}\left(\frac{x-y}{x}\right) dydx \underbrace{=}_{x\to\frac{1}{x}}$$

$$= \int_{x=\frac{1}{2}}^{1}\left(\int_{y=0}^{1} \frac{\tan^{-1}(1-yx)}{x^2} dy\right) dx \underbrace{=}_{y\to\frac{y}{x}} \int_{x=\frac{1}{2}}^{1}\left(\int_{y=0}^{x} \frac{\tan^{-1}(1-y)}{x^3} dy\right) dx \underbrace{=}_{y\to 1-y}$$

$$= \int_{x=\frac{1}{2}}^{1}\left(\int_{y=1-x}^{1} \frac{\tan^{-1}(y)}{x^3} dy\right) dx \underbrace{=}_{x\to 1-x} \int_{x=0}^{\frac{1}{2}} \frac{\left(\int_{y=x}^{1} \tan^{-1}(y)\,dy\right)}{(1-x)^3} dx =$$

$$= \int_0^{\frac{1}{2}} \frac{f(1)-f(x)}{(1-x)^3} dx \qquad Here,\ \ f(x) = \int_0^x \tan^{-1}(y)\,dy = x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2)$$

$$= \int_0^{\frac{1}{2}} \left(f(1)-f(x)\right) d\left[\frac{1}{2(1-x)^2}\right] = \left[\frac{f(1)-f(x)}{2(1-x)^2}\right]_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{f'(x)}{2(1-x)^2} dx =$$

$$= \frac{3}{2}f(1) - 2f\left(\frac{1}{2}\right) + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{\tan^{-1}(x)}{(1-x)^2} dx$$

$$Now,\quad \int_0^{\frac{1}{2}} \frac{\tan^{-1}(x)}{(1-x)^2} dx = \int_0^{\frac{1}{2}} \tan^{-1}(x)\, d\left[\frac{1}{(1-x)}\right] \underbrace{=}_{IBP} \left[\frac{\tan^{-1}(x)}{1-x}\right]_0^{\frac{1}{2}} - \int_{x=0}^{\frac{1}{2}} \frac{1}{1-x} d[\tan^{-1}(x)]$$

$$= 2\phi - \int_0^{\phi} \frac{1}{1-\tan\theta} d\theta = 2\phi - \int_0^{\phi} \frac{\cos\theta}{\cos\theta - \sin\theta} d\theta \qquad Here,\ \ \phi = \tan^{-1}\left(\frac{1}{2}\right) = \cot^{-1}(2)$$

$$= 2\phi + \int_0^{\phi} \frac{\cos\theta}{\sin\theta - \cos\theta} d\theta = 2\phi + \frac{1}{2}\int_0^{\phi} \left(\frac{\cos\theta + \sin\theta}{\sin\theta - \cos\theta} - 1\right) d\theta = \frac{3}{2}\phi + \frac{1}{2}\ln|\sin\phi - \cos\phi| =$$

$$= \frac{3}{2}\phi + \frac{1}{2}\ln\left|\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}}\right| = \frac{3}{2}\phi - \frac{1}{4}\ln 5$$

$$\Rightarrow I = \frac{3}{2}f(1) - 2f\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{3}{2}\phi - \frac{1}{4}\ln 5\right) = \frac{3}{2}f(1) - 2f\left(\frac{1}{2}\right) + \frac{3}{4}\phi - \frac{\ln 5}{8}$$

Now, $f(x) = x\tan^{-1}(x) - \frac{1}{2}\ln(1 + x^2)$

$$\Rightarrow f(1) = \frac{\pi}{4} - \frac{1}{2}\ln 2 \quad \text{and} \quad f\left(\frac{1}{2}\right) = \frac{\phi}{2} - \frac{\ln 5}{2} + \ln 2$$

$$\Rightarrow I = \frac{3}{2}\left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) - 2\left(\frac{\phi}{2} - \frac{\ln 5}{2} + \ln 2\right) + \frac{3}{4}\phi - \frac{\ln 5}{8} = \frac{3\pi}{8} - \frac{11}{4}\ln 2 - \frac{\phi}{4} + \frac{7}{8}\ln 5$$

$$\Rightarrow I = \frac{1}{8}(3\pi - 22\ln 2 - 2\cot^{-1}2 + 7\ln 5)$$