ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^\infty \frac{x\ln(1+x)}{(x^2+1)(x+1)(2x+1)}dx = \frac{1}{480}\left(144G - 19\pi^2 + 96\ln^2 2 + 36\pi\ln 2\right)$$

G is the Catalan's constant

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shobhit Jain-India

$$I = \int_0^\infty \frac{x\ln(1+x)}{(x^2+1)(x+1)(2x+1)}dx = \int_0^\infty \frac{\ln(1+x)}{10}\left(\frac{5}{1+x} - \frac{8}{1+2x} - \frac{x}{1+x^2} + \frac{3}{1+x^2}\right)dx =$$

$$= \int_0^\infty \frac{\ln(1+x)}{10}\left(\frac{5}{1+x} - \frac{8}{1+2x} - \frac{x}{1+x^2}\right)dx + \frac{3}{10}\int_0^\infty \frac{\ln(1+x)}{(x^2+1)}dx =$$

$$= \frac{1}{10}\int_0^\infty g(x)\ln(1+x)\,dx + \frac{3}{10}\int_0^\infty \frac{\ln(1+x)}{(x^2+1)}dx$$

$$here,\ \ g(x) = \frac{5}{1+x} - \frac{8}{1+2x} - \frac{x}{1+x^2}$$

$$\int_0^\infty \frac{\ln(1+x)}{(x^2+1)}dx = \int_0^1 \frac{\ln(1+x)}{(x^2+1)}dx + \int_1^\infty \frac{\ln(1+y)}{(y^2+1)}dy \underbrace{=}_{y\to\frac{1}{x}} \int_0^1 \frac{\ln(1+x)}{(x^2+1)}dx + \int_0^1 \frac{\ln\left(1+\frac{1}{x}\right)}{(x^2+1)}dx$$

$$= 2\int_0^1 \frac{\ln(1+x)}{(x^2+1)}dx + \int_0^1 \frac{\ln\left(\frac{1}{x}\right)}{(x^2+1)}dx$$

$$2\int_0^1 \frac{\ln(1+x)}{(x^2+1)}dx \underbrace{=}_{x\to\frac{1-x}{1+x}} 2\int_0^1 \frac{\ln\left(\frac{2}{1+x}\right)}{(x^2+1)}dx = \int_0^1 \frac{\ln(1+x) + \ln\left(\frac{2}{1+x}\right)}{(x^2+1)}dx = \int_0^1 \frac{\ln 2}{(x^2+1)}dx = \frac{\pi}{4}\ln 2$$

$$And,\ \int_0^1 \frac{\ln\left(\frac{1}{x}\right)}{(x^2+1)}dx = \int_0^1 \ln\left(\frac{1}{x}\right)\left(1 - x^2 + x^4 - \dots\dots\dots\right)dx = \left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots\dots\right) = G$$

$$\Rightarrow \int_0^\infty \frac{\ln(1+x)}{(x^2+1)}dx = \frac{\pi}{4}\ln 2 + G$$

$$\Rightarrow I=\frac{1}{10}\int_0^{\infty} g(x)\,ln(1+x)\,dx+\frac{3}{10}\left(\frac{\pi}{4}ln2+G\right)=\frac{1}{10}\int_0^{\infty} g(x)\,ln(1+x)\,dx+\frac{1}{480}(36\pi ln2+144G)$$

$$Now,\ \int_0^{\infty} g(x)\,ln(1+x)\,dx=\int_0^{1} g(x)\,ln(1+x)\,dx+\int_1^{\infty} g(y)\,ln(1+y)\,dy$$

$$\underbrace{=}_{y\to\frac{1}{x}}\int_0^{1} g(x)\,ln(1+x)\,dx+\int_0^{1}\frac{1}{x^2}g\left(\frac{1}{x}\right)ln\left(1+\frac{1}{x}\right)dx$$

$$=\underbrace{\int_0^{1} ln(1+x)\left(g(x)+\frac{1}{x^2}g\left(\frac{1}{x}\right)\right)dx}_{A}-\underbrace{\int_0^{1}(lnx)\left(\frac{1}{x^2}g\left(\frac{1}{x}\right)\right)dx}_{B}$$

$$\Rightarrow I=\frac{1}{10}(A-B)+\frac{1}{480}(36\pi ln2+144G).\quad Now,\ g(x)=\frac{5}{1+x}-\frac{8}{1+2x}-\frac{x}{1+x^2}$$

$$\Rightarrow \frac{1}{x^2}g\left(\frac{1}{x}\right)=\frac{5}{x(1+x)}-\frac{8}{x(x+2)}-\frac{1}{x(1+x^2)}=\frac{4}{x+2}-\frac{5}{1+x}+\frac{x}{1+x^2}$$

$$\Rightarrow\ g(x)+\frac{1}{x^2}g\left(\frac{1}{x}\right)=\frac{4}{x+2}-\frac{8}{1+2x}=\frac{-12}{(x+2)(1+2x)}$$

$$\Rightarrow A=-12\int_0^{1}\frac{ln(1+x)}{(x+2)(1+2x)}dx\underbrace{=}_{\substack{x\to\frac{1}{t}-1\\ t=\frac{1}{1+x}}}-12\int_1^{\frac{1}{2}}\frac{lnt}{(1+t)(2-t)}dt=-4\int_1^{\frac{1}{2}}\frac{lnt}{(1+t)}dt-4\int_1^{\frac{1}{2}}\frac{lnt}{(2-t)}dt$$

$$=4f(-1)-4f(2)$$

$$\Rightarrow I=\frac{2}{5}\big(f(-1)-f(2)\big)-\frac{B}{10}+\frac{1}{480}(36\pi ln2+144G)$$

$$Here\ i\ define\ f(a)=\int_1^{\frac{1}{2}}\frac{lnt}{a-t}dt=\sum_{n=1}^{\infty}\frac{1}{a^n}\int_1^{\frac{1}{2}}(lnt)t^{n-1}dt\qquad for\ |a|\geq 1$$

$$Now,\ \int_1^{\frac{1}{2}}(lnt)t^{n-1}dt=\left[(lnt)\frac{t^n}{n}-\frac{t^n}{n^2}\right]_1^{\frac{1}{2}}=-\frac{ln2}{n2^n}-\frac{1}{n^2 2^n}+\frac{1}{n^2}$$

$$\Rightarrow f(a)=\sum_{n=1}^{\infty}\frac{1}{a^n}\left(-\frac{ln2}{n2^n}-\frac{1}{n^2 2^n}+\frac{1}{n^2}\right)=(ln2)ln\left(1-\frac{1}{2a}\right)-Li_2\left(\frac{1}{2a}\right)+Li_2\left(\frac{1}{a}\right)$$

$$\Rightarrow f(-1)=(ln2)ln\left(\frac{3}{2}\right)-Li_2\left(-\frac{1}{2}\right)+Li_2(-1)=(ln2)(ln3)-ln^2 2-Li_2\left(-\frac{1}{2}\right)-\frac{\pi^2}{12}$$

$$\Rightarrow f(2)=(ln2)ln\left(\frac{3}{4}\right)-Li_2\left(\frac{1}{4}\right)+Li_2\left(\frac{1}{2}\right)=(ln2)(ln3)-2ln^2 2-Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{12}-\frac{ln^2 2}{2}$$

$$=(ln2)(ln3)-Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{12}-\frac{5}{2}ln^2 2$$

$$Now,\ \ B=\int_0^1 (lnx)\left(\frac{1}{x^2}g\left(\frac{1}{x}\right)\right)dx=\int_0^1 (lnx)\left(\frac{4}{x+2}-\frac{5}{1+x}+\frac{x}{1+x^2}\right)dx$$

$$=4\int_0^1 \frac{\frac{1}{2}lnx}{1+\frac{x}{2}}dx+5\int_0^1 \frac{ln\left(\frac{1}{x}\right)}{1+x}dx-\int_0^1 \frac{xln\left(\frac{1}{x}\right)}{1+x^2}dx$$

$$Now,\ \ Li_2(z)=-\int_0^1 \frac{zlnx}{1-zx}dx \underbrace{\Rightarrow}_{z=-\frac{1}{2}} \int_0^1 \frac{\frac{1}{2}lnx}{1+\frac{x}{2}}dx=Li_2\left(-\frac{1}{2}\right)$$

$$And\ \int_0^1 \frac{ln\left(\frac{1}{x}\right)}{1+x}dx=\int_0^1 ln\left(\frac{1}{x}\right)(1-x+x^2-\ \dots\dots\dots)dx=\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ \dots\dots\right)=\eta(2)=\frac{\pi^2}{12}$$

$$\int_0^1 \frac{xln\left(\frac{1}{x}\right)}{1+x^2}dx=\int_0^1 ln\left(\frac{1}{x}\right)(x-x^3+x^5-\ \dots\dots\dots)dx=\left(\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{6^2}-\ \dots\dots\right)=\frac{\eta(2)}{4}=\frac{\pi^2}{48}$$

$$\Rightarrow B=4Li_2\left(-\frac{1}{2}\right)+\frac{5\pi^2}{12}-\frac{\pi^2}{48}=4Li_2\left(-\frac{1}{2}\right)+\frac{19\pi^2}{48}$$

$$\Rightarrow I=\frac{2}{5}\left(f(-1)-f(2)\right)-\frac{1}{10}\left(4Li_2\left(-\frac{1}{2}\right)+\frac{19\pi^2}{48}\right)+\frac{1}{480}(36\pi ln2+144G)$$

$$\Rightarrow I=\frac{2}{5}\left(f(-1)-f(2)-Li_2\left(-\frac{1}{2}\right)\right)+\frac{1}{480}(36\pi ln2+144G-19\pi^2)$$

$$Now,\ \ f(-1)-f(2)-Li_2\left(-\frac{1}{2}\right)$$

$$=\left((ln2)(ln3)-ln^2 2-Li_2\left(-\frac{1}{2}\right)-\frac{\pi^2}{12}\right)-\left((ln2)(ln3)-Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{12}-\frac{5}{2}ln^2 2\right)-Li_2\left(-\frac{1}{2}\right)$$

$$=\frac{3}{2}ln^2 2-\frac{\pi^2}{6}+2\left(\frac{1}{2}Li_2\left(\frac{1}{4}\right)-Li_2\left(-\frac{1}{2}\right)\right)$$

$$Now\ use\ the\ identity,\ \ Li_2(z)+\ Li_2(-z)=\frac{1}{2}Li_2(z^2)\ \underbrace{\Rightarrow}_{z=\frac{1}{2}}\ Li_2\left(\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)=\frac{1}{2}Li_2\left(\frac{1}{4}\right)$$

$$\Rightarrow \frac{1}{2}Li_2\left(\frac{1}{4}\right) - Li_2\left(-\frac{1}{2}\right) = Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{ln^2 2}{2}$$

$$\Rightarrow f(-1) - f(2) - Li_2\left(-\frac{1}{2}\right) = \frac{3}{2}ln^2 2 - \frac{\pi^2}{6} + 2\left(\frac{\pi^2}{12} - \frac{ln^2 2}{2}\right) = \frac{1}{2}ln^2 2$$

$$\Rightarrow I = \frac{2}{5}\left(\frac{1}{2}ln^2 2\right) + \frac{1}{480}\left(36\pi ln2 + 144G - 19\pi^2\right)$$

$$= \frac{1}{5}ln^2 2 + \frac{1}{480}\left(36\pi ln2 + 144G - 19\pi^2\right) \Rightarrow I = \frac{1}{480}\left(144G - 19\pi^2 + 96ln^2 2 + 36\pi ln2\right)$$