ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$I=\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(xy)}{(1+x)^2(1+y)}\,dx\,dy=\frac{\log(2)}{96}\left(48G-12\pi\log(2)-5\pi^2\right)$$

Where, G is Catalan's constant

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$$I=\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(xy)}{(1+x)^2(1+y)}\,dx\,dy=\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(x)}{(1+x)^2(1+y)}\,dx\,dy+\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(y)}{(1+x)^2(1+y)}\,dx\,dy$$

$$=I_1+I_2$$

$$I_1=\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(x)}{(1+x)^2(1+y)}\,dx\,dy$$

$$=\log(2)\int_0^1\frac{\tan^{-1}(x)\log(x)}{(1+x)^2}\,dx\overset{IBP}{=}\log(2)\left\{-\frac{\pi}{4}\log(2)+\frac{1}{2}\int_0^1\frac{\log(x)}{1+x}\,dx\right.$$

$$\left.+\int_0^1\frac{\log(1+x)}{1+x^2}\,dx-\frac{1}{2}\int_0^1\frac{\log(x)}{1+x^2}\,dx-\frac{1}{2}\int_0^1\frac{x\log(x)}{1+x^2}\,dx\right\}$$

$$=\log(2)\left\{\frac{1}{2}G-\frac{\pi}{8}\log(2)-\frac{3}{16}\zeta(2)\right\}$$

$$I_2=\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(y)}{(1+x)^2(1+y)}\,dx\,dy=-\frac{\zeta(2)}{2}\int_0^1\frac{\tan^{-1}(x)}{(1+x)^2}\,dx=-\frac{1}{8}\log(2)\,\zeta(2)$$

$$I=I_1+I_2=\frac{\log(2)}{96}\left(48G-12\pi\log(2)-5\pi^2\right)$$