ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$9\Omega^2(x)+9\Omega(x)+2=0,\Omega(x)=\int_1^x\sqrt{\frac{t}{1+t^3}}dt$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo, Solution 2 by Amin Hajiyev-Azerbaijan, Solution 3 by Ravi Prakash-New Delhi-India

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo

$$\Omega(x)=\int_1^x\frac{t^{\frac{1}{2}}dt}{\sqrt{1+\left(t^{\frac{3}{2}}\right)^2}}=\left|\begin{matrix}t^{\frac{3}{2}}=u\\ \frac{2}{3}t^{\frac{1}{2}}dt=du\end{matrix}\right|=\frac{2}{3}\int_{u_1}^{u_2}\frac{du}{\sqrt{1+4^2}}=$$

$$=\frac{2}{3}\ln\left(t^{\frac{3}{2}}+\sqrt{1+t^3}\right)|_1^x=$$

$$=\frac{2}{3}\ln\left(x^{\frac{3}{2}}+\sqrt{1+x^3}\right)-\frac{2}{3}\ln(1+\sqrt{2})=\frac{2}{3}\ln\frac{x^{\frac{3}{2}}+\sqrt{1+x^3}}{1+\sqrt{2}}$$

$$\Omega(x)=\frac{-9\pm\sqrt{9}}{18}=\frac{-9\pm3}{18}=\frac{-3\pm1}{6}=\left\langle\begin{matrix}-\frac{2}{3}\\ \frac{1}{3}\end{matrix}\right.$$

1) $\ln\frac{x\sqrt{x}+\sqrt{1+x^3}}{1+\sqrt{2}}=-1,\underbrace{x\sqrt{x}+\sqrt{1+x^3}}_{\geq1}=\underbrace{\frac{(1+\sqrt{2})}{e}}_{<1}$, no solution!

2) $\ln\frac{x\sqrt{x}+\sqrt{1+x^3}}{1+\sqrt{2}}=\frac{1}{2}$

$$x\sqrt{x}+\sqrt{1+x^3}=(1+\sqrt{2})\sqrt{e}$$

Let $(1+\sqrt{2})\sqrt{e}=a$

$$\sqrt{1+x^3}=a-\sqrt{x^3}|^2$$

$$1+x^3=a^2-2a\sqrt{x^3}+x^3$$

$$2a\sqrt{x^3}=a^2-1$$

$$x^3=\frac{a^2-1}{2a}$$

$$x=\sqrt[3]{\frac{a^2-1}{2a}}$$

Solution 2 by Amin Hajiyev-Azerbaijan

$$9\Omega^2(x)+9\Omega(x)+2=0;\Omega(x)=\int_1^x\sqrt{\frac{t}{1+t^3}}dt$$

$$\left(\Omega(x)+\frac{2}{3}\right)\left(\Omega(x)+\frac{1}{3}\right)=0\to\Omega_1(x)=-\frac{2}{3}\quad\Omega_2(x)=-\frac{1}{3}$$

$$\Omega(x)=\int_1^x\frac{\sqrt{t}}{\sqrt{1+t^3}}dt\therefore\left(t^3=u\ \frac{du}{dt}=3t^2=3u^{\frac{2}{3}}\right)$$

$$\Omega(x)=\frac{1}{3}\int_1^{x^3}\frac{1}{\sqrt{u+u^2}}du\therefore\left(\sqrt{u}=v\ \frac{dv}{du}=\frac{1}{2v}\ \ v\in[x\sqrt{x};1]\right)$$

$$\Omega(x)=\frac{1}{3}\int_1^{x\sqrt{x}}\frac{2v}{\sqrt{v^2+v^4}}dv=\frac{2}{3}\int_1^{x\sqrt{x}}\frac{1}{\sqrt{1+v^2}}dv=$$

$$=\frac{2}{3}\Big|_1^{x\sqrt{x}}\sinh^{-1}(v)=\frac{2}{3}\left(\sinh^{-1}(x\sqrt{x})-\sinh^{-1}(1)\right)$$

$$\begin{cases}-\frac{2}{3}=\frac{2}{3}\left(\sinh^{-1}(x\sqrt{x})-\sinh^{-1}(1)\right)\\-\frac{1}{3}=\frac{2}{3}\left(\sinh^{-1}(x\sqrt{x})-\sinh^{-1}(1)\right)\end{cases}\quad\begin{cases}\sinh^{-1}(x\sqrt{x})=\sinh^{-1}(1)-1\\\sinh^{-1}(x\sqrt{x})=\sinh^{-1}(1)-\frac{1}{2}\end{cases}$$

Answer:

$$x_1=\sinh^{\frac{2}{3}}(\sinh^{-1}(1)-1)$$

$$x_2=\sinh^{\frac{2}{3}}\left(\sinh^{-1}(1)-\frac{1}{2}\right)$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\Omega(x)=\int_1^x\sqrt{\frac{t}{1+t^3}}dt=\int_1^x\frac{t^{\frac{1}{2}}}{\sqrt{1+\left(t^{\frac{3}{2}}\right)^2}}dt$$

Put $t^{\frac{3}{2}} = u \Rightarrow \frac{3}{2}t^{\frac{1}{2}}dt = du$

$$\therefore \Omega(x) = \frac{2}{3}\int_1^{x^{\frac{3}{2}}} \frac{du}{\sqrt{1+u^2}} = \frac{2}{3}\log\left(u+\sqrt{1+u^2}\right)\Big]_1^{x^{\frac{3}{2}}}$$

$$= \frac{2}{3}\left[\log\left(x^{\frac{3}{2}}+\sqrt{1+x^3}\right)-\log\left(1+\sqrt{2}\right)\right] \quad (1)$$

Now, $9\Omega(x)^2 + 9\Omega(x) + 2 = 0 \Rightarrow (3\Omega(x)+1)(3\Omega(x)+2) = 0$

$$\Rightarrow \Omega(x) = -\frac{1}{3} \text{ or } \Omega(x) = -\frac{2}{3} \quad (2)$$

Now,

$$\frac{2}{3}\left[\log\left(x^{\frac{3}{2}}+\sqrt{1+x^3}\right)-\log\left(1+\sqrt{2}\right)\right] = -\frac{1}{3}, -\frac{2}{3}$$

[from (1) and (2)]

$$\Rightarrow \log\left(x^{\frac{3}{2}}+\sqrt{1+x^3}\right)-\log\left(1+\sqrt{2}\right) = -\frac{1}{2}, -1$$

$$\Rightarrow \frac{x^{\frac{3}{2}}+\sqrt{1+x^3}}{1+\sqrt{2}} = \frac{1}{\sqrt{e}}, \frac{1}{e} \Rightarrow \sqrt{1+x^3}+x^{\frac{3}{2}} = \frac{(\sqrt{2}+1)}{\sqrt{e}}, \frac{\sqrt{2}+1}{e} \quad (3)$$

$$\Rightarrow \frac{(1+x^3)-x^3}{\sqrt{1+x^3}-x^{\frac{3}{2}}} = \frac{\sqrt{2}+1}{\sqrt{e}}, \frac{\sqrt{2}+1}{e} \Rightarrow \sqrt{1+x^3}-x^{\frac{3}{2}} = \frac{\sqrt{e}}{\sqrt{2}+1}, \frac{e}{\sqrt{2}+1} \quad (4)$$

Subtracting (4) from (3) we get

$$2x^{\frac{3}{2}} = \frac{\sqrt{2}+1}{\sqrt{e}} - \frac{\sqrt{e}}{\sqrt{2}+1}, \frac{\sqrt{2}+1}{e} - \frac{e}{\sqrt{2}+1}$$

$$\Rightarrow 2x^{\frac{3}{2}} = \frac{\left(\sqrt{2}+1\right)^2 - e}{\sqrt{e}\left(\sqrt{2}+1\right)}, \frac{\left(\sqrt{2}+1\right)^2 - e^2}{\left(\sqrt{2}+1\right)e}$$

But $\frac{\left(\sqrt{2}+1\right)^2 - e^2}{\left(\sqrt{2}+1\right)e} < 0$

therefore

$$x^{\frac{3}{2}} = \frac{\left(\sqrt{2}+1\right)^2 - e}{\sqrt{e}\left(2\sqrt{2}+2\right)} \Rightarrow x = \left[\frac{\left(\sqrt{2}+1\right)^2 - e}{\sqrt{e}\left(2\sqrt{2}+2\right)}\right]^{\frac{2}{3}}$$