ROMANIAN MATHEMATICAL MAGAZINE

$$f:\mathbb{R}\to\mathbb{R},\ f(f(x))+f(x)=-x,\ \forall x\in\mathbb{R}$$

Find:

$$\Omega=\int_0^{\pi} f\left(f\left(f\left(\frac{1}{2+\cos x}\right)\right)\right)dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Samed Ahmedov-Azerbaijan

$$f(f(x))+f(x)=-x,\qquad f\left(f(f(x))\right)+f(f(x))=-f(x)$$

$$f\left(f(f(x))\right)+f(f(x))+f(x)=0,\qquad f\left(f(f(x))\right)-x=0$$

$$f\left(f\left(f\left(\frac{1}{2+\cos x}\right)\right)\right)-\frac{1}{2+\cos x}==\Rightarrow f\left(f(f(x))\right)=\frac{1}{2+\cos x}$$

$$\Omega=\int_0^{\pi}\frac{1}{2+\cos x}dx,\qquad \cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\Omega=\int_0^{\pi}\frac{1}{2+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}dx=\int_0^{\pi}\frac{1}{\frac{2+2\tan^2\frac{x}{2}+1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}dx=\int_0^{\pi}\frac{1+\tan^2\frac{x}{2}}{3+\tan^2\frac{x}{2}}dx$$

$$1+\tan^2\frac{x}{2}=\sec^2\frac{x}{2}\Rightarrow\Omega=\int_0^{\pi}\frac{\sec^2\frac{x}{2}}{3+\tan^2\frac{x}{2}}dx$$

$$\tan\frac{x}{2}=m\ \ x=0,m=0.x=\pi m=+\infty dx=\frac{2dm}{\sec^2\frac{x}{2}}$$

$$\Omega=\int_0^{+\infty}\frac{\sec^2\frac{x}{2}}{3+m^2}\cdot\frac{2}{\sec^2\frac{x}{2}}dm=\int_0^{+\infty}\frac{2}{\left(\sqrt{3}\right)^2+m^2}dm$$

$$\Omega=\lim_{\eta\to+\infty}\int_0^{\eta}\frac{2}{\left(\sqrt{3}\right)^2+m^2}dm=\lim_{\eta\to+\infty}\left(\frac{2}{\sqrt{3}}\arctan\frac{m}{\sqrt{3}}\right)\Bigg|_0^{\eta}=\lim_{\eta\to+\infty}\left(\frac{2}{\sqrt{3}}\arctan\frac{\eta}{\sqrt{3}}-0\right)=$$

$$=\frac{\pi\sqrt{3}}{3}=\frac{\pi}{\sqrt{3}},\qquad \Omega=\frac{\pi}{\sqrt{3}}$$

Solution 2 by Adrian Popa-Romania

$$f\left(f\left(f\left(\frac{1}{2+\cos x}\right)\right)\right)=-f\left(\frac{1}{2+\cos x}\right)-f\left(f\left(\frac{1}{2+\cos x}\right)\right)=$$

$$=-f\left(\frac{1}{2+\cos x}\right)-\left(-\frac{1}{2+\cos x}-f\left(\frac{1}{2+\cos x}\right)\right)=\frac{1}{2+\cos x}$$

$$\Omega=\int_0^{\pi}\frac{1}{2+\cos x}dx=\int_0^{\pi}\frac{1}{2+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}dx=\int_0^{\pi}\frac{1+\tan^2\frac{x}{2}}{3+\tan^2\frac{x}{2}}dx$$

$$\tan\frac{x}{2}=t\Rightarrow\frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx=dt$$

$$\left(1+\tan^2\frac{x}{2}\right)dx=2dt$$

$$x=0\Rightarrow t=0$$

$$x=\pi\Rightarrow t\to\infty$$

$$\Omega=\int_0^{\infty}\frac{2dt}{3+t^2}=2\cdot\frac{1}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}}\Big|_0^{\infty}=\frac{2}{\sqrt{3}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{3}}$$

Solution 3 by Ankush Kumar Parcha-India

We have, $f:\mathbb{R}\to\mathbb{R},(f\circ f)x+f(x)=x,\forall x\in\mathbb{R}$

$$\overset{x\to f(x)}{\Longrightarrow}(f\circ f\circ f)x+(f\circ f)x=-f(x)\underset{above}{\overset{From}{\Longrightarrow}}(f\circ f\circ f)(x)=x\qquad(1)$$

$$\underset{(1)}{\overset{Equation}{\Longrightarrow}}\int_0^{\pi}(f\circ f\circ f)\left(\frac{1}{2+\cos x}\right)dx=\int_0^{\pi}\frac{dx}{2+\cos(x)}$$

$$\overset{x\to\pi-x}{\Longrightarrow}\int_0^{\pi}\frac{dx}{2+\cos(\pi-x)}=\int_0^{\pi}\frac{dx}{2-\cos(x)}$$

$$\Rightarrow 2\int_0^{\pi}\frac{dx}{4-\cos^2(x)}=2\int_0^{\frac{\pi}{2}}\frac{dx}{4-\cos^2(x)}+2\underbrace{\int_{\frac{\pi}{2}}^{\pi}\frac{dx}{4-\cos^2(x)}}_{x\to\frac{\pi}{2}+x}$$

$$\overset{\because\cos\left(x+\frac{\pi}{2}\right)=-\sin(x)}{\Longrightarrow}2\int_0^{\frac{\pi}{2}}\frac{dx}{4-\cos^2(x)}+2\int_0^{\frac{\pi}{2}}\frac{dx}{4-\sin^2(x)}$$

$$\overset{\because \tan^2(x)+1=\sec^2(x)}{\Longrightarrow} 2\int_0^{\frac{\pi}{2}} \frac{\sec^2(x)}{4\tan^2(x)+3}dx + 2\int_0^{\frac{\pi}{2}} \frac{\sec^2(x)}{3\tan^2(x)+4}dx$$

$$\overset{\tan(x)\to x}{\Longrightarrow} 2\int_0^{\infty} \frac{dx}{4x^2+3} + 2\int_0^{\infty} \frac{dx}{3x^2+4} = \left(\frac{\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{x\sqrt{3}}{2}\right)}{\sqrt{3}}\right)_0^{\infty}$$

$$= \frac{\pi}{2\sqrt{3}} + \frac{\pi}{2\sqrt{3}} \Rightarrow \int_0^{\pi} (f\circ f\circ f)\left(\frac{1}{2+\cos x}\right)dx = \frac{\pi}{\sqrt{3}}$$

Solution 4 by Ravi Prakash-New Delhi-India

$$f\big(f(x)\big) + f(x) = -x \qquad \textbf{(1)}$$

$$\Rightarrow f\Big(f\big(f(x)\big)\Big) + f\big(f(x)\big) = -f(x) \Rightarrow f\Big(f\big(f(x)\big)\Big) + f\big(f(x)\big) + f(x) = 0$$

$$\Rightarrow f\Big(f\big(f(x)\big)\Big) - x = 0 \quad \text{[using (1)]} \Rightarrow f\Big(f\big(f(x)\big)\Big) = x$$

$$\therefore \Omega = \int_0^{\infty} \frac{dx}{2+\cos x} = \int_0^{\pi} \frac{dx}{1+2\cos^2\left(\frac{x}{2}\right)} = \int_0^{\pi} \frac{\sec^2\left(\frac{x}{2}\right)dx}{\tan^2\left(\frac{x}{2}\right)+3}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)\Bigg]_0^{\pi} = \left(\frac{2}{\sqrt{3}}\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{\sqrt{3}}$$