

Find:

$$\Omega = \int_0^2 \int_0^2 \max \left(\min \left(x, \frac{1}{y}, \frac{xy+1}{x} \right) \right) dx dy$$

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Case 1:

$$x \leq \frac{1}{y} \Rightarrow xy \leq 1$$

$$\min \left(x, \frac{1}{y}, \frac{xy+1}{x} \right) = \min \left(x, \frac{xy+1}{x} \right) \leq \sqrt{x \cdot \frac{xy+1}{x}} = \sqrt{xy+1} \leq \sqrt{1+1} = \sqrt{2}$$

Case 2:

$$x \geq \frac{1}{y} \Rightarrow xy \geq 1 \Rightarrow \frac{1}{xy} \leq 1$$

$$\begin{aligned} \min \left(x, \frac{1}{y}, \frac{xy+1}{x} \right) &= \min \left(\frac{1}{y}, \frac{xy+1}{x} \right) \leq \sqrt{\frac{1}{y} \cdot \frac{xy+1}{x}} = \sqrt{\frac{xy+1}{xy}} = \\ &= \sqrt{1 + \frac{1}{xy}} \leq \sqrt{1+1} = \sqrt{2} \end{aligned}$$

$$\max \left(\min \left(x, \frac{1}{y}, \frac{xy+1}{x} \right) \right) = \sqrt{2}$$

$$\Omega = \int_0^2 \int_0^2 \max \left(\min \left(x, \frac{1}{y}, \frac{xy+1}{x} \right) \right) dx dy = \int_0^2 \int_0^2 \sqrt{2} dx dy = 4\sqrt{2}$$