ROMANIAN MATHEMATICAL MAGAZINE

$$f,g:R\to R,\ f(1)=3,\ g(1)=2,\ xf(y)+yf(x)=2f(xy)$$

$$xg(y)+yg(x)=2g(xy),\ \forall\, x,y\in R$$

Find :

$$\Omega=\int_0^1 \frac{f\left(\frac{\sinh(x)}{3}\right)}{f\left(\frac{\sinh(x)}{3}\right)+f\left(\frac{\cosh(x)}{2}\right)}dx$$

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Solution by Shirvan Tahirov-Azerbaijan

$$xf(y)+yf(x)=2f(xy)\Rightarrow\ xf(1)+f(x)=2f(x)\Rightarrow f(x)=xf(1)=3x$$

$$xg(y)+yg(x)=2g(xy)\Rightarrow xg(1)+2g(x)=2g(x)\Rightarrow g(x)=xg(1)=2x$$

$$f\left(\frac{\sinh(x)}{3}\right)=\sinh(x)\ ,\qquad f\left(\frac{\cosh(x)}{2}\right)=\cosh(x)$$

$$\int_0^1 \frac{f\left(\frac{\sinh(x)}{3}\right)}{f\left(\frac{\sinh(x)}{3}\right)+f\left(\frac{\cosh(x)}{2}\right)}dx=\int_0^1\frac{\sinh(x)}{\sinh(x)+\cosh(x)}dx=\int_0^1\frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x-e^{-x}}{2}+\frac{e^x+e^{-x}}{2}}dx=$$

$$\int_0^1\frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x-e^{-x}+e^x+e^{-x}}{2}}dx=\int_0^1\frac{\frac{e^x-e^{-x}}{2}}{\frac{2e^x}{2}}dx=\int_0^1\frac{e^x-e^{-x}}{2e^x}dx=\left(\frac{1}{4e^2}+\frac{1}{2}-\frac{1}{4}\right)=\left(\frac{1}{4e^2}+\frac{1}{4}\right)$$