ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1 x^n\sqrt{1-x^2}\,dx$$

Proposed by Marin Chirciu-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$I=\int_0^1 x^n\sqrt{1-x^2}\,dx \overset{x\to x^2}{\cong} \frac{1}{2}\int_0^1 x^{\frac{n}{2}-\frac{1}{2}}(1-x)^{\frac{1}{2}}dx$$

$$I=\frac{1}{2}\beta\left(\frac{n+1}{2},\frac{3}{2}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}=\frac{1}{2}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}=\frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}$$

$$I=\int_0^1 x^n\sqrt{1-x^2}\,dx=\frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}$$