ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$\int_0^{\infty}\frac{dx}{x^4+x^2+1}=\frac{\pi}{2\sqrt{3}}$$

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$$I=\int_0^{\infty}\frac{dx}{x^4+2x^2\cos(2a)+1}\Bigg]_{a=\frac{\pi}{6}\text{ and }x=\frac{1}{t}}=\int_0^{\infty}\frac{t^2dt}{t^4+2t^2\cos(2a)+1}$$

$$2I=\int_0^{\infty}\frac{dx}{x^4+2x^2\cos(2a)+1}+\int_0^{\infty}\frac{x^2dx}{x^4+2x^2\cos(2a)+1}$$

$$=\int_0^{\infty}\frac{(x^2+1)dx}{x^4+2x^2\cos(2a)+1}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{x^4+2x^2\cos(2a)+1}$$

$$I=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{x^4+2x^2\cos(2a)+1}=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{x^4+2x^2(1-2\sin^2(a))+1}$$

$$=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{x^4+2x^2+1-4x^2\sin^2(a)}=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{(x^2+1)^2-4x^2\sin^2(a)}$$

$$=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{(x^2+1-2x\sin(a))(x^2+1+2x\sin(a))}$$

$$\text{We know that }A=\frac{1}{4}\int_{-\infty}^{\infty}\frac{-2x\sin(a)dx}{(x^2+1-2x\sin(a))(x^2+1+2x\sin(a))}=0$$

Then let's clarify the sum of A+I

$$A+I=I=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{(x^2+1-2x\sin(a))(x^2+1+2x\sin(a))}$$

$$+\frac{1}{4}\int_{-\infty}^{\infty}\frac{-2x\sin(a)dx}{(x^2+1-2x\sin(a))(x^2+1+2x\sin(a))}$$

$$=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2-2x\sin(a)+1)dx}{(x^2+1-2x\sin(a))(x^2+1+2x\sin(a))}$$

$$=\frac{1}{4}\int_{-\infty}^{\infty}\frac{dx}{x^2+1+2x\sin(a)}$$

$$I=\frac{1}{4}\int_{-\infty}^{\infty}\frac{dx}{x^2+\sin^2(a)+\cos^2(a)+2x\sin(a)}=\frac{1}{4}\int_{-\infty}^{\infty}\frac{dx}{(x+\sin(a))^2+\cos^2(a)}$$

$$=\frac{1}{4\cos(a)}\operatorname{arctg}\left(\frac{x+\sin(a)}{\cos(a)}\right)\Bigg]_{-\infty}^{\infty}=\frac{\pi}{4\cos(a)}\Bigg]_{a=\frac{\pi}{6}}=\frac{\pi}{2\sqrt{3}}$$