ROMANIAN MATHEMATICAL MAGAZINE

Find all values $\alpha \in [-2024, 2024]$ such that:

$$\int_{\alpha}^{2024} (|2025x| - x^2 + 2026x)\,dx \le \alpha^2 + 2025 \ (1)$$

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Put $m = 2024$ for easy demonstration. We consider 2 cases:

Case 1. $-m \le \alpha < 0$

We have $\alpha < 0 < m$, so we have:

$$\int_{\alpha}^{m} (|(m+1)x| - x^2 + (m+2)x)\,dx = \int_{0}^{m} \left((2m+3)x - x^2\right)\,dx + \int_{\alpha}^{0} (x - x^2)\,dx$$

$$\to \int_{\alpha}^{m} (|(m+1)x| - x^2 + (m+2)x)\,dx = \frac{m^2}{6}(4m+9) + \frac{1}{6}\alpha^2(2\alpha - 3)$$

So we can rewrite the inequality (1) as:

$$\frac{m^2}{6}(4m+9) + \frac{1}{6}\alpha^2(2\alpha - 3) \le \alpha^2 + m + 1$$

$$\to \frac{\alpha^3}{3} - \frac{3\alpha^2}{2} + \frac{2m^3}{3} + \frac{3m^2}{2} - m - 1 \ge 0$$

Consider the function $f(\alpha) = \frac{\alpha^3}{3} - \frac{3\alpha^2}{2} + \frac{2m^3}{3} + \frac{3m^2}{2} - m - 1$ in range $[-m, 0)$

We have $f'(\alpha) = \alpha^2 - 3\alpha = \alpha(\alpha - 3) > 0$.So the function $f(\alpha)$ is a increasing function in range $[-m, 0) \to f(\alpha) \ge f(-m) \to f(\alpha) \ge \frac{m^3}{3} - m - 1 > 0$.So for all $-m \le \alpha < 0$, the inequality (1) is satisfied.

Case 2. $0 \le \alpha \le m$

We have $0 \le \alpha \le m$, so we have:

$$\int_{\alpha}^{m} (|(m+1)x| - x^2 + (m+2)x)\,dx = \int_{\alpha}^{m} (x - x^2)\,dx = \frac{\alpha^3}{3} - \frac{\alpha^2}{2} + \frac{m^2}{2} - \frac{m^3}{3}$$

So we can rewrite the inequality (1) as:

$$\frac{\alpha^3}{3}-\frac{\alpha^2}{2}+\frac{m^2}{2}-\frac{m^3}{3}\le \alpha^2+m+1$$

$$\to \frac{\alpha^3}{3}-\frac{3\alpha^2}{2}-\frac{m^3}{3}+\frac{m^2}{2}-m-1\le 0$$

Consider the function $g(\alpha)=\frac{\alpha^3}{3}-\frac{3\alpha^2}{2}-\frac{m^3}{3}+\frac{m^2}{2}-m-1$ **in range** $[0,m]$

We have $g'(\alpha)=\alpha^2-3\alpha=\alpha(\alpha-3)$**. So** $g'(\alpha)=0\to\alpha=3$

We have:

$$\begin{cases} g(0)=-\frac{m^3}{3}+\frac{m^2}{2}-m-1<0 \\ g(3)=-\frac{m^3}{3}+\frac{m^2}{2}-m<0 \\ g(m)=-m^2-m-1<0 \end{cases}$$

$\to g(\alpha)\le 0\ \forall\ \alpha\in[0,m]$**. So for all** $0\le\alpha\le m$**, the inequality** (1) **is satisfied.**

In conclusion, all real numbers $\alpha\in[-2024,2024]$ **satisfy the inequality** (1)**.**