ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \int_0^1 \int_0^1 \frac{x^2 \ln(y)\ln(x)\ln(1+x^2)}{(1+y)^2}\,dxdy$$

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$$\Omega = \int_0^1 \int_0^1 \frac{x^2 \ln(y)\ln(x)\ln(1+x^2)}{(1+y)^2}\,dxdy$$

$$= \int_0^1 \frac{\ln(y)}{(1+y)^2}dy \times \int_0^1 x^2\ln(x)\ln(1+x^2)\,dx = K \times M$$

$$K = \int_0^1 \frac{\ln(y)}{(1+y)^2}dy$$

$$\left\{\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x};\ \frac{\partial}{\partial x}\sum_{n=0}^{\infty}(-1)^n x^n = -\frac{1}{(1+x)^2}\ ;\ -\sum_{n=0}^{\infty}(-1)^n x^{n-1} = \frac{1}{(1+x)^2}\right\}$$

$$K = -\sum_{n=1}^{\infty}(-1)^n n\int_0^1 y^{n-1}\ln(y)\,dy = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} = --\ln(2)$$

$$M = \int_0^1 x^2\ln(x)\ln(1+x^2)\,dx = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{2n+2}\ln(x)\,dx =$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n(2n+3)^2} = \frac{1}{9}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} - \frac{2}{9}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+3} - \frac{2}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+3)^2} =$$

$$= \frac{\ln(2)}{9} + \frac{2}{9}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+5} - \frac{16}{27} + \frac{2}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}$$

$$\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2} = \sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{2n+4}\text{dx} = \int_0^1\frac{x^4}{1+x^2}dx = \right\}$$

$$\left\{= \int_0^1 (x^2-1)\text{dx} + \int_0^1 \frac{1}{1+x^2}dx = \frac{1}{3} - 1 + \frac{\pi}{4} = \frac{\pi}{4} - \frac{2}{3}\right\}$$

$$\text{M} = -\frac{\ln(2)}{9} + \frac{\pi}{18} - \frac{4}{27} - \frac{16}{27} + \frac{\text{G}}{3} = -\frac{\ln(2)}{9} + \frac{\pi}{18} - \frac{20}{27} + \frac{2\text{G}}{3}$$

answer: $\Omega = K \times M = \frac{ln^2(2)}{9} - \frac{\pi\ln(2)}{18} + \frac{20\ln(2)}{27} - \frac{2\ln(2)}{3}G$

Note: G - Catalan's constant