ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_1^{\infty}\frac{\ln^2(x)}{(1+x+x^2)^2}dx = \frac{16\sqrt{3}}{729}\pi^3-\frac{7}{81}\pi^2-\frac{2}{27}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{4}{27}\psi^{(1)}\left(\frac{2}{3}\right)$$

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$$\int_1^{\infty}\frac{\ln^2(x)}{(1+x+x^2)^2}dx=\int_1^{\infty}\frac{(x-1)^2}{(x^3-1)^2}\ln^2(x)\ dx$$

$$\Rightarrow (x\to 1/x)\int_0^1\frac{(x-x^2)^2}{(1-x^3)^2}\ln^2(x)\ dx$$

$$\Rightarrow (x^3\to x)\ \frac{1}{27}\int_0^1\frac{\left(1-\sqrt[3]{x}\right)^2}{(1-x)^2}\ln^2(x)\ dx\Rightarrow$$

$$\frac{1}{27}\sum_{n\epsilon\mathbb{N}} n\ \int_0^1\left(x^{n-1}+x^{n-\frac{1}{3}}-2x^{n-\frac{2}{3}}\right)\ \ln^2(x)\ dx\Rightarrow$$

$$\frac{2}{27}\sum_{n\epsilon\mathbb{N}} n\ \left(\frac{1}{n^3}+\frac{27}{(3n+2)^3}-\frac{54}{(3n+1)^3}\right)=$$

$$=\frac{2}{27}\zeta(2)+\frac{2}{27}\zeta\left(2,\frac{2}{3}\right)-\frac{4}{81}\zeta\left(3,\frac{2}{3}\right)-\frac{4}{27}\zeta\left(2,\frac{1}{3}\right)+\frac{4}{81}\zeta\left(3,\frac{1}{3}\right)\Rightarrow$$

$$\frac{\pi^2}{81}-\frac{2}{27}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{4}{27}\psi^{(1)}\left(\frac{2}{3}\right)-\frac{2}{27}\left[\psi^{(1)}\left(\frac{1}{3}\right)+\psi^{(1)}\left(\frac{2}{3}\right)\right]+\frac{2}{81}\left[\psi^{(2)}\left(\frac{2}{3}\right)-\psi^{(2)}\left(\frac{1}{3}\right)\right]$$

$$\Rightarrow \frac{\pi^2}{81}-\frac{2}{27}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{4}{27}\psi^{(1)}\left(\frac{2}{3}\right)-\frac{2}{27}\ \frac{\pi^2}{\sin^2(\pi/3)}+\frac{4\pi^3}{81}\ \frac{\cot(\pi/3)}{\sin^2(\pi/3)}$$

$$\Rightarrow\ \int_1^{\infty}\frac{\ln^2(x)}{(1+x+x^2)^2}\ dx=\frac{16\sqrt{3}}{729}\pi^3-\frac{7}{81}\pi^2-\frac{2}{27}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{4}{27}\psi^{(1)}\left(\frac{2}{3}\right)$$