Find:

$$\int_0^1\int_0^1 x^3\arctan^3(1-x^2)\ln(\ln^4(y))\,dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Bui Hong Suc-Vietnam

$$\therefore\ \gamma=-\int_0^\infty e^{-x}\ln x\,dx=-\int_0^1\ln(-\ln x)\,dx$$

$$\therefore\ \int_0^{\frac{\pi}{4}}\ln\cos x\,dx=-\frac{\pi}{4}\ln 2+\frac{G}{2}$$

$$\therefore\ \ln\cos x=-\ln 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos 2nx$$

$$\therefore\ \int_0^{\frac{\pi}{4}}x\ln\cos x\,dx=\int_0^{\frac{\pi}{4}}x\left(-\ln 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos 2nx\right)dx$$

$$=-\frac{x^2}{2}\ln 2\,\Big|_0^{\frac{\pi}{4}}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}x\cos 2nxdx$$

$$=-\frac{\pi^2\ln 2}{32}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\pi n\sin\frac{n\pi}{2}+2\cos\frac{n\pi}{2}-2}{8n^2}=-\frac{\pi^2\ln 2}{32}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\frac{n\pi}{2}}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\frac{n\pi}{2}}{n^3}=-\frac{\pi^2\ln 2}{32}+\frac{1}{4}\cdot\frac{-3}{4}\zeta(3)-\frac{\pi}{8}\cdot(-G)-\frac{1}{4}\cdot\frac{1}{8}\cdot\frac{-3}{4}\zeta(3)=\frac{16\pi G-21\zeta(3)-4\pi^2\ln 2}{128}$$

$$\Omega=\int_0^1\int_0^1 x^3\arctan^3(1-x^2)\ln(\ln^4(y))\,dxdy=\int_0^1 x^3\arctan^3(1-x^2)\,dx\int_0^1\ln(-\ln(y))^4\,dy$$

$$=4\int_0^1 x^2\,x\arctan^3(1-x^2)\,dx\int_0^1\ln(-\ln(y))\,dy=4A.(-\gamma)=-4\gamma A$$

$$\therefore\ A=\int_0^1 x^2\,x\arctan^3(1-x^2)\,dx=\frac{1}{2}\int_0^1(1-v)\arctan^3 v\,dv$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{(1-\tan x)}{\cos^2 x}x^3\,dx=\frac{1}{2}\left(\int_0^{\frac{\pi}{4}}x^3\,d\tan x-\frac{1}{2}\int_0^{\frac{\pi}{4}}x^3\,d\tan^2 x\right)$$

$$*)J=\int_0^{\frac{\pi}{4}}x^3\,d\tan x=x^3\tan x\,\Big|_0^{\frac{\pi}{4}}+3\int_0^{\frac{\pi}{4}}x^2\,d\ln\cos x=$$

$$=\frac{\pi^3}{64}+3x^2\ln\cos x\,\Big|_0^{\frac{\pi}{4}}-6\int_0^{\frac{\pi}{4}}x\ln\cos x\,dx$$

$$=\frac{\pi^3}{64}-\frac{3\pi^2}{32}\ln 2-6\int_0^{\frac{\pi}{4}}x\ln\cos x\,dx=\frac{\pi^3}{64}-\frac{3\pi^2}{32}\ln 2-6\,\frac{16\pi G-21\zeta(3)-4\pi^2\ln 2}{128}$$

$$=\frac{\pi^3+6\pi^2\ln 2+63\zeta(3)-48\pi G}{64}$$

$$*)K=\int_0^{\frac{\pi}{4}}x^3\,d\tan^2 x=x^3\tan^2 x\,\Big|_0^{\frac{\pi}{4}}-3\int_0^{\frac{\pi}{4}}x^2\tan^2 x\,dx=\frac{\pi^3}{64}-3\int_0^{\frac{\pi}{4}}x^2\left(\frac{1}{\cos^2 x}-1\right)dx$$

$$=\frac{\pi^3}{64}+3\int_0^{\frac{\pi}{4}}x^2\,dx-3\int_0^{\frac{\pi}{4}}x^2\,d\tan x=\frac{\pi^3}{64}+x^3\Big|_0^{\frac{\pi}{4}}-3\left(x^2\tan x\Big|_0^{\frac{\pi}{4}}-2\int_0^{\frac{\pi}{4}}x\tan x\,dx\right)$$

$$=\frac{\pi^3}{64}+\frac{\pi^3}{64}-3\left(\frac{\pi^2}{16}+2\int_0^{\frac{\pi}{4}}x\,d\ln\cos x\right)$$

$$=\frac{\pi^3}{32}-\frac{3\pi^2}{16}-6\left(x\ln\cos x\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\ln\cos x\,dx\right)$$

$$=\frac{\pi^3}{32}-\frac{3\pi^2}{16}+\frac{3\pi}{4}\ln 2+6\int_0^{\frac{\pi}{4}}\ln\cos x\,dx=\frac{\pi^3}{32}-\frac{3\pi^2}{16}+\frac{3\pi}{4}\ln 2+6\left(-\frac{\pi}{4}\ln 2+\frac{G}{2}\right)$$

$$=\frac{\pi^3}{32}-\frac{3\pi^2}{16}-\frac{3\pi}{4}\ln 2+3G$$

Hence :

$$A=\frac{1}{2}\left(J-\frac{1}{2}K\right)=\frac{1}{2}\left(\frac{\pi^3+6\pi^2\ln 2+63\zeta(3)-48\pi G}{64}-\frac{1}{2}\left(\frac{\pi^3}{32}-\frac{3\pi^2}{16}-\frac{3\pi}{4}\ln 2+3G\right)\right)$$

$$=\frac{3}{2}\left(\frac{21\zeta(3)-16G(\pi+2)+2\pi(\pi+\pi\ln 2+4\ln 2)}{64}\right)$$

$$\textit{Then}:\Omega=-4\gamma A=-4\gamma\frac{3}{2}\left(\frac{21\zeta(3)-16G(\pi+2)+2\pi(\pi+\pi\ln 2+4\ln 2)}{64}\right)$$

$$=-\frac{3\gamma}{32}\left(21\zeta(3)-16G(\pi+2)+2\pi(\pi+\pi\ln 2+4\ln 2)\right)$$

Note : G "Catalan's" constant
ζ (3) "Apery's constant"
γ "Euler-Mascheroni constant"