ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^\infty\int_0^\infty \frac{\cos(x+y)}{\sqrt{x\sqrt{y}}}\,dxdy = \frac{(\sqrt{2}-2)}{4}\sqrt{\pi}\csc\left(\frac{\pi}{8}\right)\Gamma\left(\frac{3}{4}\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Ankush Kumar Parcha-India

$$\int_0^\infty\int_0^\infty \frac{\cos(x+y)}{\sqrt{x\sqrt{y}}}\,dxdy \Rightarrow \mathfrak{R}\int_0^\infty\int_0^\infty \frac{e^{i(x+y)}}{\sqrt{x\sqrt{y}}}\,dxdy$$

$$\left(\because \frac{\Gamma(z)}{x^z} = \int_0^\infty t^{z-1}\,e^{-xt}dt,\ \mathfrak{R}(z) > 0 \wedge \mathfrak{R}(x) > 0\right)$$

$$\Rightarrow \Gamma(1/2)\Gamma(3/4)\mathfrak{R}\left(i^{5/4}\right) \Rightarrow \sqrt{\pi}\Gamma(3/4)\mathfrak{R}\left(e^{i5\pi/8}\right)$$

$$\Rightarrow \sqrt{\pi}\Gamma(3/4)\cos(5\pi/8) \Rightarrow -\sqrt{\pi}\Gamma(3/4)\sin^2(\pi/8)\csc(\pi/8)$$

$$\Rightarrow \int_0^\infty\int_0^\infty \frac{\cos(x+y)}{\sqrt{x\sqrt{y}}}\,dxdy = \frac{(\sqrt{2}-2)}{4}\sqrt{\pi}\csc\left(\frac{\pi}{8}\right)\Gamma\left(\frac{3}{4}\right)$$