ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1\int_0^1 (\ln\left(\ln(1+x)+arctan^2(1-y)\right)dxdy=$$

$$=\frac{\pi^2}{16}+\frac{\pi}{4}\ln(2)+2\ln\ln(2)+\gamma-\mathrm{li}(2)-G$$

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$$\int_0^1\int_0^1 (\ln\left(\ln(1+x)+arctan^2(1-y)\right)dxdy$$

$$\underbrace{\int_0^1\int_0^1 (\ln\ln(1+x)\,dxdy}_{:=\xi_1}+\underbrace{\int_0^1\int_0^1 arctan^2(1-y)dxdy}_{:=\xi_2}$$

$$\xi_1=\int_0^1\int_0^1 (\ln\left(\ln(1+x)+arctan^2(1-y)\right)dxdy$$

$$\overset{I.B.P}{\Longrightarrow}\left(ln ln(1+x)\int\frac{dx}{dy}xdx\right)_0^1-\int_0^1\frac{x}{(1+x)\ln(1+x)}dx\overset{1+x\to x}{\Longrightarrow}ln ln(2)-\int_1^2\frac{dx}{x}+$$

$$+\underbrace{\int_1^2\frac{dx}{x ln(x)}}_{\ln(x)\to x}\underset{(1)}{\overset{Note\ section}{\underbrace{\Longrightarrow}}}\ln\ln(2)+\underbrace{Li(1)}_{li(1)-li(2)}+\ln\ln(2)-\lim_{x\to 0}\ln(x)$$

$$\underset{(2)}{\overset{Note\ section}{\underbrace{\Longrightarrow}}}2ln ln(2)+\gamma+\lim_{x\to 1}\ln\ln(x)-li(2)-\lim_{x\to 0}\ln(x)$$

$$\Rightarrow \xi_1=\int_0^1\int_0^1 \ln(1+x)\,dxdy-2ln ln(2)+\gamma-li(2)$$

$$\xi_2=\int_0^1\int_0^1 arctan^2(1-y)dxdy\overset{1-y\to x}{\Longrightarrow}\int_0^1 \mathrm{arctan}^2(\mathrm{x})\mathrm{dx}$$

$$\overset{I.B.P}{\Longrightarrow}\left(arctan^2(x)\int\frac{dx}{dy}dx\right)_0^1-2\int_0^1\frac{\mathrm{x}\tan^{-1}(x)}{1+x^2}dx$$

$$\overset{I.B.P}{\Longrightarrow}\frac{\pi^2}{16}-\left(\tan^{-1}(x)\int\frac{dx}{dy}\ln(1+x^2)\,dx\right)_0^1+\underbrace{\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx}_{x\to\tan(x)}$$

$$\overset{tan^2(x)+1-sec^2(x)}{\Longrightarrow}\frac{\pi^2}{16}-\frac{\pi}{4}\ln(2)-2\int_0^{\frac{\pi}{4}}\ln\cos(\mathrm{x})dx\underset{(3)}{\overset{Note\ section}{\underbrace{\Longrightarrow}}}\frac{\pi^2}{16}-\frac{\pi}{4}\ln(2)+$$

$$+2\ln(2)\int_0^{\frac{\pi}{4}}dx+2\sum_{n\in N}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2\mathrm{nx})dx$$

$$\Rightarrow \frac{\pi^2}{16} - \frac{\pi}{4}\ln(2) + \sum_{n\in N}\frac{(-1)^n}{n}\left(\frac{\sin(2nx)}{n}\right)\Big|_0^{\pi/4} \xrightarrow{n-2n+1} \frac{\pi^2}{16} + \frac{\pi}{4}\ln(2) + \sum_{n\in N\cup\{0\}}\frac{(-1)^{n+1}}{(2n+1)^2}$$

$$\underbrace{\overset{Note\ section}{\Longrightarrow}}_{(4)} \xi_2 = \int_0^1\int_0^1 arctan^2(1-y)dxdy = \frac{\pi^2}{16} + \frac{\pi}{4}\ln(2) - \mathrm{G}$$

Put the values of ξ_1 and ξ_2 in equation-(1)

$$\int_0^1\int_0^1 \left(\ln\left(\ln(1+x) + arctan^2(1-y)\right)dxdy = \frac{\pi^2}{16} + \frac{\pi}{4}\ln(2) + 2\ln\ln(2) + \gamma - \mathrm{li}(2) - \mathrm{G}\right.$$

Note: G − **Catalans constant**

$\gamma - Euler - Mascheroni\ constant$